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# Non-parabolic prehomogeneous vector spaces and exceptional Lie algebras

Hubert Rubenthaler

*Institut de Recherche Mathématique Avancée, Université Louis Pasteur et CNRS, 7 rue René Descartes, 67084 Strasbourg cedex, France*

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**Abstract**

Among the classification of irreducible regular and reduced prehomogeneous vector spaces, there are only a few one which are not of parabolic type. In this paper we investigate this family of prehomogeneous vector and show that they nevertheless all occur inside the exceptional Lie algebras and are in some sense close to the parabolic family. Moreover we give a Lie theoretical description of their relative invariants and the corresponding characters.


## 1. Introduction

The theory of prehomogeneous vector spaces (abbreviated *PV*) was introduced by Mikio Sato [21], mainly to produce new kind of zeta functions. This was developed and generalized by several authors. Without any seek of completeness let us mention [3,4, 11,20,22–25]. For a more complete bibliography the reader is referred to the book of T. Kimura [8].

Let us recall very briefly the basic definitions. We start with a connected algebraic group $G$. Except in Lemma 4.2, the group $G$ is always supposed to be reductive. We denote also by $G$ the group of complex points. We suppose also that we are given a finite

*E-mail address:* rubenth@math.u-strasbg.fr.

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dimensional irreducible representation $\rho$ of $G$ in a complex vector space $V$. The basic assumptions we make are the following:

- $G$ has a Zariski open orbit $\Omega$ in $V$, $((G, \rho, V)$ or $(G, V)$ is then called a prehomogeneous vector space). The elements in the open orbit are called *generic* elements.
- There exists on $V$ a non-trivial polynomial $P$ which is relatively invariant under $G$, and whose Hessian $H(P)(x)$ is non-zero on $\Omega$ $((G, \rho, V)$ is then called *regular*).

It can be easily shown that the property of $(G, \rho, V)$ to be a *PV* is in fact an infinitesimal condition. Let $\mathfrak{g}$ be the Lie algebra of $G$ and let $d\rho$ be the derived representation of $\mathfrak{g}$. Then $(G, \rho, V)$ is a *PV* if and only if there exists $v \in V$ such that the map

$$\mathfrak{g} \to V, \quad X \mapsto d\rho(X)v,$$

is surjective (see [22]). Therefore we will often call $(\mathfrak{g}, V)$ a prehomogeneous vector space if the preceding condition is satisfied. The set of such $v$'s is exactly the open orbit.

Then it can be proved that the contragredient representation $(G, \rho^*, V^*)$ is again a regular prehomogeneous vector space, whose fundamental relative invariant is denoted by $P^*$. If we assume further that $(G, \rho, V)$ is defined over $\mathbb{Q}$ (and some other technical conditions, see [20,23,25] or [8]) then one can associate global zeta function to this datum, and prove a functional equation for it.

The regular irreducible *PV*'s have been classified by Sato and Kimura [22] up to an equivalence relation called the castling transform. These authors gave a list of so-called reduced representatives in each castling class. The precise definition of a castling transform and of a reduced representative is not needed here.

On the other hand, the author introduced a new class of irreducible regular *PV*'s, the so called irreducible regular prehomogenous vector spaces of parabolic type [14–16]. These are called parabolic because they are closely related to well behaved maximal parabolic subalgebras of simple Lie algebras (see Section 2 below).

By comparing the classifications in [22] and in [14] (or [15,16]), one observes that most of the irreducible regular *PV*'s are of parabolic type. Except for the so-called trivial *PV*'s (see [8, p. 223]), the irreducible reduced non-parabolic *PV*'s are listed in Table 1. (In this table we only indicate the infinitesimal *PV*, $\Lambda_1$ stands for the natural representation of $\mathfrak{gl}(p)$ on $\mathbb{C}^p \simeq V(p)$, $\Lambda_2$ stands for the irreducible 7-dimensional representation of the exceptional Lie algebra $G_2$, *spin* stands for the Spin representation of the corresponding orthogonal Lie algebra $\mathfrak{o}(k)$, and $V(p)$ always denotes a complex vector space of dimension $p$.) We will show that these *PV*'s have nevertheless close relations to the parabolic family.

The preceding, short list however splits into three parts corresponding to increasing complexity of their structure. The *PV*'s (1-1), (1-2) and (1-3) have very simple explanations in terms of parabolic subalgebras of simple algebras, as well as their relative invariants (see Section 4 below). The *PV*'s (2-1) and (2-2) have the same simple explanation but the construction of their relative invariant is more involved (see Sections 5 and 6 below). More precisely, the construction of the relative invariants of (2-1) and (2-2) in terms of Lie structure needs the construction of the relative invariants for some non-irreducible *PV*'s

Table 1

| | |
|---|---|
| (1-1) | $(\mathfrak{gl}(2)\times\mathfrak{o}(7), \Lambda_1\otimes spin, V(2)\otimes V(8))$ |
| (1-2) | $(\mathfrak{gl}(3)\times\mathfrak{o}(7), \Lambda_1\otimes spin, V(3)\otimes V(8))$ |
| (1-3) | $(\mathfrak{gl}(1)\times\mathfrak{o}(11), \Lambda_1\otimes spin, V(1)\otimes V(32))$ |
| (2-1) | $(\mathfrak{gl}(1)\times\mathfrak{o}(9), \Lambda_1\otimes spin, V(1)\otimes V(16))$ |
| (2-2) | $(\mathfrak{gl}(1)\times G_2, \Lambda_1\otimes\Lambda_2, V(1)\otimes V(7))$ |
| (3-1) | $(\mathfrak{gl}(2)\times G_2, \Lambda_1\otimes\Lambda_2, V(2)\otimes V(7))$ |

of parabolic type (see Theorem 5.2 , Corollary 5.3, Theorem 6.2 and Corollary 6.3). The *PV* (3-1) has a much more involved interpretation in terms of Lie algebras. This interpretation takes place into the exceptional Lie algebra $E_6$ and is strongly related to the triality principle. It will be developed in Section 8, as well as a Lie theoretical description of its fundamental relative invariant. Before that, in Section 2 we will recall the definition of *PV*'s of parabolic type and in Section 3 we will recall the definition of admissible and $C$-admissible subalgebras. In Section 7 we will prove a general result on *PV*'s of commutative parabolic type which will be used in Section 8.

## 2. *PV*'s of parabolic type

At this point a brief summary of the theory of *PV*'s of parabolic type is needed.

Let $\mathfrak{g}$ be a simple complex Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and denote by $\Sigma$ the set of roots of $(\mathfrak{g},\mathfrak{h})$. As usually, for $\alpha\in\Sigma$, we denote by $H_\alpha$ the corresponding co-root in $\mathfrak{h}$. We fix once and for all a system of simple roots $\Psi$ for $\Sigma$. We denote by $\Sigma^+$ (respectively $\Sigma^-$) the corresponding set of positive (respectively negative) roots in $\Sigma$. Let $\theta$ be a subset of $\Psi$ and let us make the standard construction of the parabolic subalgebra $\mathfrak{p}_\theta\subset\mathfrak{g}$ associated to $\theta$. As usual we denote by $\langle\theta\rangle$ the set of all roots which are linear combinations of elements in $\theta$, and put $\langle\theta\rangle^\pm=\langle\theta\rangle\cap\Sigma^\pm$.

Set

$$\mathfrak{h}_\theta=\theta^\perp=\bigl\{X\in\mathfrak{h}\mid\alpha(X)=0\ \forall\alpha\in\theta\bigr\},\qquad \mathfrak{h}(\theta)=\sum_{\alpha\in\theta}\mathbb{C}H_\alpha,$$
$$\mathfrak{l}_\theta=\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}_\theta)=\mathfrak{h}\oplus\sum_{\alpha\in\langle\theta\rangle}\mathfrak{g}^\alpha,\qquad \mathfrak{n}_\theta^\pm=\sum_{\alpha\in\Sigma^\pm\setminus\langle\theta\rangle^\pm}\mathfrak{g}^\alpha.$$

Then $\mathfrak{p}_\theta=\mathfrak{l}_\theta\oplus\mathfrak{n}_\theta^+$ is called the standard parabolic subalgebra associated to $\theta$. There is also a standard $\mathbb{Z}$-grading of $\mathfrak{g}$ related to these data. Define $H_\theta$ to be the unique element of $\mathfrak{h}_\theta$ satisfying the linear equations

$$\alpha(H_\theta)=0\quad\forall\alpha\in\theta\qquad\text{and}\qquad\alpha(H_\theta)=2\quad\forall\alpha\in\Psi\setminus\theta.$$

The before mentioned grading is just the grading obtained from the eigenspace decomposition of $\operatorname{ad}H_\theta$:

$$d_p(\theta)=\bigl\{X\in\mathfrak{g}\mid[H_\theta,X]=2pX\bigr\}.$$

Then we obtain easily:

$$\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} d_p(\theta), \qquad \mathfrak{l}_\theta = d_0(\theta), \qquad \mathfrak{n}_\theta^+ = \sum_{p\geqslant 1} d_p(\theta), \qquad \mathfrak{n}_\theta^- = \sum_{p\leqslant -1} d_p(\theta).$$

It is known that $(\mathfrak{l}_\theta, d_1(\theta))$ is a prehomogeneous vector space (in fact all the spaces $(\mathfrak{l}_\theta,\, d_p(\theta))$ with $p \neq 0$ are prehomogeneous, but there is no loss of generality if we only consider $(\mathfrak{l}_\theta,\, d_1(\theta))$). These spaces have been called prehomogeneous vector spaces of parabolic type [14]. There are in general neither irreducible nor regular. But they are of particular interest, because in the parabolic context, the group (or more precisely its Lie algebra $\mathfrak{l}_\theta$) and the space (here $d_1(\theta)$) of the *PV* are embedded into a rich structure, namely the simple Lie algebra $\mathfrak{g}$. For example the derived representation of the *PV* is just the adjoint representation of $\mathfrak{l}_\theta$ on $d_1(\theta)$. Moreover the Lie algebra $\mathfrak{g}$ also contains the dual *PV*, namely $(\mathfrak{l}_\theta, d_{-1}(\theta))$.

It may be worthwhile to notice also that $d_1(\theta) = \sum_{\beta\in\sigma_1} \mathfrak{g}^\beta$, where $\sigma_1$ is the set of roots which belong to the set $(\Psi \setminus \theta) + \mathbb{Z}\theta$, where $\mathbb{Z}\theta$ is the $\mathbb{Z}$-span of $\theta$.

As these *PV*'s are in one to one correspondence with the subsets $\theta \subset \Psi$, we make the convention to describe them by the mean of the following weighted Dynkin diagram:

**Definition 2.1.** The diagram of the *PV* $(\mathfrak{l}_\theta, d_1(\theta))$ is the Dynkin diagram of $(\mathfrak{g}, \mathfrak{h})$ (or $\Sigma$), where the vertices corresponding to the simple roots of $\Psi \setminus \theta$ are circled (see an example below).

This very simple classification by means of diagrams contains nevertheless some immediate and interesting informations concerning the *PV* $(\mathfrak{l}_\theta, d_1(\theta))$ (for all these facts, see [14,15] or [16]):

- The Dynkin diagram of $\mathfrak{l}'_\theta = [\mathfrak{l}_\theta, \mathfrak{l}_\theta]$ (i.e., the semi-simple part of the Lie algebra of the group) is the Dynkin diagram of $\mathfrak{g}$ where we have removed the circled vertices and the arrows connected to these vertices.
- In fact as a Lie algebra $\mathfrak{l}_\theta = \mathfrak{l}_\theta{}' \oplus \mathfrak{h}_\theta$ and $\dim \mathfrak{h}_\theta =$ the number of circled vertices.
- The number of irreducible components of the representation $(\mathfrak{l}_\theta,\, d_1(\theta))$ is also equal to the number of circled roots. More precisely, if $\alpha$ is a (simple) circled root, then any non-zero root vector $X_\alpha \in \mathfrak{g}^\alpha$ generates an irreducible $\mathfrak{l}_\theta$-module $V_\alpha$, and $d_1(\theta) = \bigoplus_{\alpha\in\Psi\setminus\theta} V_\alpha$ is the decomposition of $d_1(\theta)$ into irreducibles.

In fact the decomposition of the representation $(\mathfrak{l}_\theta, d_1(\theta))$ into irreducibles can also be described by using the eigenspace decomposition with respect to $\mathrm{ad}(\mathfrak{h}_\theta)$. Let me explain this. For each $\alpha \in \mathfrak{h}^*$, let $\overline{\alpha}$ be the restriction of $\alpha$ to $\mathfrak{h}_\theta$ and define

$$\mathfrak{g}^{\overline{\alpha}} = \big\{ X \in \mathfrak{g} \mid \forall H \in \mathfrak{h}_\theta,\ [H, X] = \overline{\alpha}(H)X \big\}.$$

Then $\mathfrak{g}^{\bar{0}} = \mathfrak{l}_\theta$ and for $\alpha \in \Psi \setminus \theta$, we have $V_\alpha = \mathfrak{g}^{\overline{\alpha}}$. Hence we can write

$$d_1(\theta) = \bigoplus_{\alpha\in\Psi\setminus\theta} \mathfrak{g}^{\overline{\alpha}}.$$

Moreover one can notice (always for $\alpha \in \Psi \setminus \theta$) that $V_\alpha = \mathfrak{g}^{\overline{\alpha}} = \sum_{\beta \in \sigma_1^\alpha} \mathfrak{g}^\beta$, where $\sigma_1^\alpha$ is the set of roots which belong to $\alpha + \mathbb{Z}\theta$.

- Moreover one can directly read the highest weight of $V_\alpha$ from the diagram.

The highest weight of $V_\alpha$ relatively to the Borel subalgebra $\mathfrak{b}_\theta^- = \mathfrak{h} \oplus \sum_{\alpha \in \langle\theta\rangle^-} \mathfrak{g}^\alpha$ is $\overline{\alpha} = \alpha|_{\mathfrak{h}(\theta)}$. Let $\omega_\beta$ $(\beta \in \theta)$ be the fundamental weights of $\mathfrak{l}'_\theta$ (i.e., the dual basis of $(H_\beta)_{\beta\in\theta}$). For each circled root $\alpha$ (i.e., for each $\alpha \in \Psi \setminus \theta$), let $J_\alpha = \{(\beta_i)\}$ be the set of roots in $\theta$ (= non-circled) which are connected to $\alpha$ in the diagram. From elementary diagram considerations we know that $J_\alpha$ may be empty and that there are always less than 3 roots in $J_\alpha$.

If $J_\alpha = \emptyset$, then $V_\alpha$ is the trivial one dimensional representation of $\mathfrak{l}_\theta$.

If $J_\alpha \neq \emptyset$, then $\overline{\alpha} = \sum_{i \in J_\alpha} c_i \omega_{\beta_i}$ where $c_i = -\alpha(H_{\beta_i})$ and $\alpha(H_{\beta_i})$ can be computed as follows:

$$\text{(R)} \quad \begin{cases} \text{if } \|\alpha\| \leqslant \|\beta_i\|, \text{ then } \alpha(H_{\beta_i}) = -1; \\ \text{if } \|\alpha\| > \|\beta_i\| \text{ and if } \alpha \text{ and } \beta_i \text{ are connected by } j \text{ arrows } (1 \leqslant j \leqslant 3), \\ \qquad \text{then } \alpha(H_{\beta_i}) = -j. \end{cases}$$

- The diagram of $(\mathfrak{l}_\theta, d_1(\theta))$, together with a table of roots allows also to describe the decomposition of the representation $(\mathfrak{l}_\theta, d_p(\theta))$ (which is also a *PV*) into irreducibles.

For this purpose let $R$ denote the set of restrictions of the roots of $\Sigma$ to $\mathfrak{h}_\theta$. For $\overline{\alpha} \in R$, let us define the $\theta$-height of $\overline{\alpha}$ by

$$h_\theta(\overline{\alpha}) = \frac{1}{2}\alpha(H_\theta).$$

It is easy to see that $h_\theta(\overline{\alpha})$ is also the sum of the coefficients of $\alpha$ with respect to the elements of $\Psi \setminus \theta$. Then

$$d_p(\theta) = \sum_{h_\theta(\overline{\alpha})=p} \mathfrak{g}^\alpha = \sum_{\Omega_p} \mathfrak{g}^{\overline{\alpha}}$$

where $\Omega_p = \{\overline{\alpha} \in R \mid h_\theta(\overline{\alpha}) = p\}$. Then it can be proved that the decomposition

$$d_p(\theta) = \sum_{\Omega_p} \mathfrak{g}^{\overline{\alpha}}$$

is the decomposition into $\mathfrak{l}_\theta$-irreducible representations. This of course is coherent with the decomposition of $d_1(\theta)$ given above.

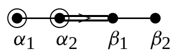
As an example define $d_{\text{top}}(\theta)$ to be the space $d_{p_0}(\theta)$ where $p_0$ is maximal among the $p$'s such that $d_p(\theta) \neq \{0\}$ (hence $\mathfrak{g} = \sum_{-p_0}^{p_0} d_p(\theta)$). Similarly we put $d_{-\text{top}}(\theta) = d_{-p_0}(\theta)$. Let $\delta$ be the highest root in $\Sigma$. Then we have

$$d_{\text{top}}(\theta) = \mathfrak{g}^{\overline{\delta}} \quad \text{and} \quad d_{-\text{top}}(\theta) = \mathfrak{g}^{-\overline{\delta}}.$$

Hence $d_{\text{top}}(\theta)$ and $d_{-\text{top}}(\theta)$ are irreducible.

Let us illustrate this with an example.

**Example 2.2.** Consider the following diagram:




The preceding diagram is the diagram of a *PV* of parabolic type inside $\mathfrak{g} \simeq F_4$. Here we have

$$\theta = \{\beta_1, \beta_2\} \quad \text{and} \quad \Psi \setminus \theta = \{\alpha_1, \alpha_2\}.$$

The Lie algebra $\mathfrak{l}_\theta$ is isomorphic to $A_2 \oplus \mathfrak{h}_\theta$ where $\dim \mathfrak{h}_\theta =$ number of circled roots $= 2$. As $J_{\alpha_1} = \emptyset$, the representation of $\mathfrak{l}'_\theta$ on $V_{\alpha_1}$ is the trivial representation. Hence the action of $\mathfrak{l}_\theta$ on $V_{\alpha_1}$ reduces to the character of $\mathfrak{h}_\theta$ given by the restriction of the root $\alpha_1$ to $\mathfrak{h}_\theta$. On the other hand we have $J_{\alpha_2} = \{\beta_1\}$. Therefore $V_{\alpha_2}$ is the irreducible $A_2$-module with highest weight $2\omega_1$, where $\{\omega_1, \omega_2\}$ is the set of fundamental weights of $A_2$ corresponding to $\beta_1$ and $\beta_2$. Again the action of $\mathfrak{h}_\theta$ on $V_{\alpha_2}$ is scalar with eigenvalue the restriction of $\alpha_2$ to $\mathfrak{h}_\theta$.

One can prove [14] that the *PV* of parabolic type $(\mathfrak{l}_\theta, d_1(\theta))$ is irreducible if and only if $\mathfrak{p}_\theta$ is a maximal parabolic subalgebra, i.e., if and only if $\Psi \setminus \theta$ is reduced to a single root $\alpha_1$. Among the irreducible *PV*'s of parabolic type there is an important class, namely the class of *PV*'s of commutative parabolic type. We say that an irreducible *PV* $(\mathfrak{l}_\theta, d_1(\theta))$ is of commutative type if the Lie algebra $\mathfrak{n}_\theta^+$ is commutative. This is equivalent to say that $d_1(\theta) = \mathfrak{n}_\theta^+$ or to say that the single root $\alpha_1$ in $\Psi \setminus \theta$ has coefficient one in the highest root of $\Sigma$.

There is also the following criterion of regularity:

**Theorem 2.3.** *Suppose that the PV* $(\mathfrak{l}_\theta, d_1(\theta))$ *is irreducible* (*i.e.*, $|\Psi \setminus \theta| = 1$). *Then* $(\mathfrak{l}_\theta, d_1(\theta))$ *is regular, if and only if there exist* $X \in d_1(\theta)$, *and* $Y \in d_{-1}(\theta)$ *such that* $(Y, H_\theta, X)$ *is an* $\mathfrak{sl}_2$*-triple* [14–16].

Using the preceding result (and the classification of a special kind of $\mathfrak{sl}_2$-triples) it is now easy to classify the *PV*'s of parabolic type which are irreducible and regular. They are classified by a list of the "weighted" Dynkin diagram of $\mathfrak{g}$, where the root $\alpha_1$ in the discussion above is circled. This classification appears first in [14] (see also [15] and [16]). Of course not every root $\alpha_1 \in \Psi$ corresponds to a regular irreducible *PV* of parabolic type. Among these *PV*'s there is an important subclass, namely the regular irreducible *PV* of commutative parabolic type, which is in one to one correspondance with simple Jordan algebras over $\mathbb{C}$. The *PV* of commutative type, as well as their real forms have been intensively studied.

For convenience, in the sequel of the paper, we will refer to the list $\mathbf{L}_r$ of all irreducible regular *PV* of parabolic type or to the list $\mathbf{L}_c$ of irreducible regular *PV* of commutative parabolic type. As said before, these both lists can be found in [14] or [15].

As the list $\mathbf{L}_c$ is rather short and as it will be usefull in the sequel we give it Fig. 1.

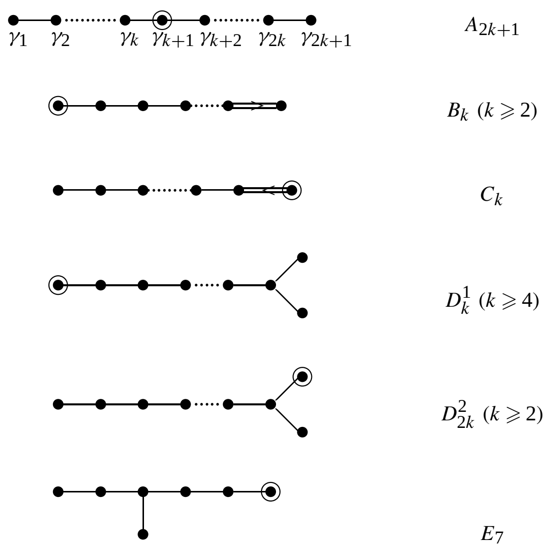



Fig. 1. List $\mathbf{L}_c$.

## 3. Admissible subalgebras

Admissible subalgebras of simple Lie algebras were introduced in [17]. They play a fundamental role in the classification of dual pairs in reductive Lie algebras [18] and in the classification of some Lie algebras which are graded by roots systems [12,13].

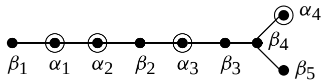
As we will need such subalgebras, we will briefly describe their construction. Details can be found in [17].

Lets us start with a *PV* of parabolic type $(\mathfrak{l}_\theta, d_1(\theta))$ and let $D$ be the coresponding weighted Dynkin diagram. If $\alpha$ is a circled root in $D$ (i.e., $\alpha \in \Psi \setminus \theta$) then we define the irreducible component $D_\alpha$ of $D$ in the following manner:

**Definition 3.1.** Let $D$ be the weighted Dynkin diagram of a *PV* of parabolic type $(\mathfrak{l}_\theta, d_1(\theta))$. Let $\alpha$ be a circled root in $D$ (i.e., $\alpha \in \Psi \setminus \theta$) then the irreducible component $D_\alpha$ is the connected component (in the sense of diagrams) of $\theta \cup \{\alpha\}$ which contains $\alpha$ (see the example below).

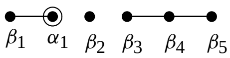
Then $D_\alpha$ is a weighted Dynkin diagram with only one circled root, namely $\alpha$. Hence it corresponds to an irreducible *PV* of parabolic type $\mathfrak{g}(\alpha)$ which is the Lie subalgebra of $\mathfrak{g}$ corresponding to $D_\alpha$. Let us denote by $\Psi_\alpha$ the roots in $D_\alpha$, and by $\theta_\alpha$ the non-circled roots in $D_\alpha$ ($\Psi_\alpha = \theta_\alpha \cup \{\alpha\}$). Of course $\Psi_\alpha$ is a set of simple roots of $\mathfrak{g}(\alpha)$, with respect to the Cartan subalgebra generated by the root vectors $H_\beta$ with $\beta \in \Psi_\alpha$.

**Example 3.2.** Consider the following weighted diagram in $D_9$

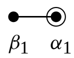
$D =$ 


where $\theta = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}$ and $\Psi \setminus \theta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

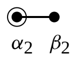
We have

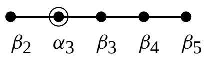
$\theta \cup \{\alpha_1\} =$ 

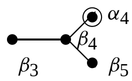

Therefore the irreducible subdiagram associated to $\alpha_1$ is given by:

$D_{\alpha_1} =$ 


Similarly the other irreducible subdiagrams of $D$ are the following:

$D_{\alpha_2} =$ 


$D_{\alpha_3} =$ 


$D_{\alpha_4} =$ 


Concretely the meaning of these subdiagrams is as follows. As explained before, in the preceding example the representation $(\mathfrak{l}_\theta, d_1(\theta))$ splits into four irreducible components $d_1(\theta) = V_{\alpha_1} \oplus V_{\alpha_2} \oplus V_{\alpha_3} \oplus V_{\alpha_4}$ under the $\mathfrak{l}_\theta$-action. In fact only a part of $\mathfrak{l}'_\theta$ acts non-trivially on $V_{\alpha_i}$, and this part corresponds precisely to the non-circled roots in $D_{\alpha_i}$. For example only the $\mathfrak{sl}_2$-type part corresponding to the root $\beta_1$ acts on $V_{\alpha_1}$. The corresponding representation is the *PV* of parabolic type associated to the diagram $D_{\alpha_1}$.

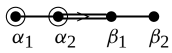
**Definition 3.3** (*cf.* [17]). The subset $\theta \subset \Psi$ or its weighted Dynkin diagram $D$ are called admissible if for every $\alpha \in \Psi \setminus \theta$, the irreducible subdiagram $D_\alpha$ corresponds to an irreducible regular *PV* of parabolic type. In other words $D$ is admissible if for every $\alpha \in \Psi \setminus \theta$, the subdiagram $D_\alpha$ belongs to the list $\mathbf{L}_r$ mentioned above.

Similarly $\theta$ or $D$ are called $C$-admissible if for every $\alpha \in \Psi \setminus \theta$, the subdiagram $D_\alpha$ corresponds to an irreducible regular *PV* of commutative parabolic type, in other words $D$ is $C$-admissible if for every $\alpha \in \Psi \setminus \theta$, the subdiagram $D_\alpha$ belongs to the sublist $\mathbf{L}_c$ mentioned above.

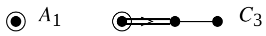
Suppose now we are given an admissible subset $\theta \subset \Psi$. Then by Theorem 2.3 , there exists an $\mathfrak{sl}_2$-triple $(Y_{\overline{\alpha}}, H_{\overline{\alpha}}, X_{\overline{\alpha}})$ inside $\mathfrak{g}$, for every $\alpha \in \Psi \setminus \theta$. The notation $Y_{\overline{\alpha}}, X_{\overline{\alpha}}$ is coherent with the usual root vectors notation, because for every $H \in \mathfrak{h}_\theta$, we have: $[H, Y_{\overline{\alpha}}] = -\overline{\alpha}(H)Y_{\overline{\alpha}}$ and $[H, X_{\overline{\alpha}}] = \overline{\alpha}(H)X_{\overline{\alpha}}$. But of course these vectors $Y_{\overline{\alpha}}$ and $X_{\overline{\alpha}}$ are not root vectors in general.

**Theorem 3.4** (cf. [17]). *If $\theta$ is admissible then the family of $\mathfrak{sl}_2$-triples $(Y_{\overline{\alpha}}, H_{\overline{\alpha}}, X_{\overline{\alpha}})$ $(\alpha \in \Psi \setminus \theta)$ generates a simple subalgebra $\mathfrak{g}_\theta$ of $\mathfrak{g}$, which is called the admissible subalgebra of $\mathfrak{g}$ associated to the admissible subset $\theta$. If moreover $\theta$ is $C$-admissible then $\mathfrak{g}_\theta$ is called $C$-admisssible.*

**Example 3.5.** Let us consider again the same diagram as in Example 2.2:




This diagram is $C$-admissble because its irreducible components




belong to the list $\mathbf{L}_c$. It can be proved (see [17]) that in this example the corresponding $C$-admissible subalgebra of $F_4$ is of type $G_2$.

**Example 3.6.** Another example (which is very instructive in my opinion) is the following. Any simple Lie algebra $\mathfrak{g}$, is admissible inside itself and corresponds to the $C$-admissible subset $\theta = \emptyset \subset \Psi$. The empty subset is effectively $C$-admissible because it correspond to a weighted Dynkin diagram where all roots are circled and hence all irreducible subdiagrams are of type


and belong to the list $\mathbf{L}_c$. In this situation of course the $\mathfrak{sl}_2$-triples which generate the admissible subalgebra are the usual triples $(Y_\alpha, H_\alpha, X_\alpha)$ where $\alpha \in \Psi$ which are well known to generate $\mathfrak{g}$.

## 4. The cases (1-1), (1-2) and (1-3)

**Definition 4.1.** Let $(G, V)$ be any representation of a group $G$ on a vector space $V$. For $x \in V$, we denote by $G_x$ the isotropy subgroup of $x$ in $G$:

$$G_x = \{g \in G \mid g.x = x\}.$$

Similarly, if $(\mathfrak{g}, V)$ is any representation of a Lie algebra on a vector space $V$, we denote by $\mathfrak{g}_x$ the isotropy subalgebra of $x$ in $\mathfrak{g}$:

$$\mathfrak{g}_x = \{X \in \mathfrak{g} \mid X.x = 0\}.$$

Of course if $G$ is a Lie group, if $(G, V)$ a finite dimensional representation of $G$ and if $(\mathfrak{g}, V)$ is the corresponding derived representation, then $\mathfrak{g}_x$ is the Lie algebra of $G_x$.

Let us first make the following easy (and well known) observation, whose proof is left to the reader.

**Lemma 4.2.** *Let $(G, V)$ be a PV (here $G$ is not supposed to be reductive). Suppose that $V = V_1 \oplus V_2$ where $V_1$ and $V_2$ are non-trivial $G$-invariant subspaces of $V$. Let $I = I_1 + I_2$*

*($I_i \in V_i$) be a generic element in $V$. Then the $G_{I_1}$-orbit of $I_2$ is open in $V_2$, hence $(G_{I_1}, V_2)$ is a PV and $I_2$ is generic for it. Moreover we have $(G_{I_1})_{I_2} = G_{(I_1+I_2)}$. Similarly, at the infinitesimal level we have*

$$\{X.I_2 \mid X \in \mathfrak{g}_{I_1}\} = V_2 \quad \textit{and} \quad (\mathfrak{g}_{I_1})_{I_2} = \mathfrak{g}_{(I_1+I_2)}.$$

*The subgroup $G_{I_1}$ (respectively the subalgebra $\mathfrak{g}_{I_1}$) is then called the partial generic isotropy sugroup (respectively the partial generic isotropy subalgebra) of $(G, V)$ (respectively $(\mathfrak{g}, V)$) relatively to $I_1$.*

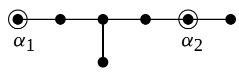
*Moreover remark that if $P$ is a relative invariant for $(G, V)$, then the rational function $Q$ defined on $V_2$ by $Q(X_2) = P(I_1 + X_2)$ $(X_2 \in V_2)$ is a relative invariant of $(G_{I_1}, V_2)$ and the character of $Q$ is the restriction to $G_{I_1}$ of the character of $P$.*

Suppose now we are given a weighted Dynkin diagram, corresponding to a subset $\theta \subset \Psi$. Denote by $\{\alpha_1, \alpha_2, \ldots, \alpha_k\} = \Psi \setminus \theta$ the set of circled roots. Let $X_{\overline{\alpha}_i}$ be a generic element in $\mathfrak{g}^{\overline{\alpha}_i} = V_{\alpha_i}$. We will denote by $(\mathfrak{l}_\theta)_{X_{\overline{\alpha}_i}}$ the centralizer of $X_{\overline{\alpha}_i}$ in $\mathfrak{l}_\theta$, which is also the partial generic isotropy relatively to $X_{\overline{\alpha}_i}$. From Lemma 4.2 we know that $((\mathfrak{l}_\theta)_{X_{\overline{\alpha}_i}}, \bigoplus_{j\neq i} V_{\alpha_j})$ is a *PV*. We call this *PV the partial isotropy PV of* $(\mathfrak{l}_\theta, d_1(\theta))$ *relatively to* $\alpha_i$ (it depends in fact on $X_{\overline{\alpha}_i}$, but up to isomorphism it depends only on $\alpha_i$).

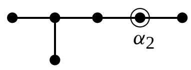
We have the following result.

**Theorem 4.3.**

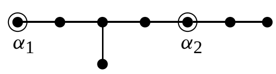
(1) *The PV* $(\mathfrak{gl}(2) \times \mathfrak{o}(7), \Lambda_1 \otimes spin, V(2) \otimes V(8))$ *is the partial isotropy of*



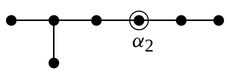

*relatively to $\alpha_1$. Its fundamental relative invariant is the same as the one corresponding to the irreducible regular subdiagram*



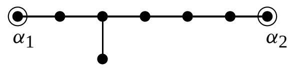

(2) *The PV* $(\mathfrak{gl}(3) \times \mathfrak{o}(7), \Lambda_1 \otimes spin, V(3) \otimes V(8))$ *is the partial isotropy of*



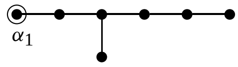

*relatively to $\alpha_1$. Its fundamental relative invariant is the same as the one corresponding to the irreducible regular subdiagram*



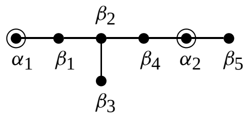
(3) *The PV* $(\mathfrak{gl}(1) \times \mathfrak{o}(11), \Lambda_1 \otimes \mathit{spin}, V(1) \otimes V(32))$ *is the partial isotropy of*




*relatively to* $\alpha_2$. *Its fundamental relative invariant is the same as the one corresponding to the irreducible regular subdiagram*




**Proof.** (1) Here $\mathfrak{l}'_\theta \simeq D_4 \times A_1$. Let us number the roots as follows

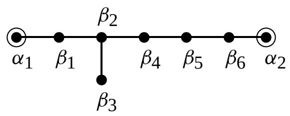



and denote by $(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5)$ the fundamental weights of $D_4 \times A_1$ corresponding to $(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$. From above we know that $\mathfrak{l}_\theta \simeq D_4 \times A_1 \times \mathfrak{h}_\theta$ where $\dim \mathfrak{h}_\theta = 2$ and that the corresponding *PV* of parabolic type decomposes into two irreducibles: $d_1(\theta) = V_{\alpha_1} \oplus V_{\alpha_2}$. Moreover the representation $(\mathfrak{l}'_\theta, V_{\alpha_1})$ is $(D_4, \omega_1)$ (the piece $A_1$ does not act) and the representation $(\mathfrak{l}'_\theta, V_{\alpha_2})$ is $(D_4 \times A_1, \omega_4 \otimes \omega_5)$ where of course $\omega_5$ is the natural 2-dimensional representation of $A_1 \simeq \mathfrak{sl}_2$ (see rule (R) in Section 2). One knows that $D_4 \simeq \mathfrak{o}(8)$ has three non-equivalent irreducible representations of dimension 8, namely $\omega_1, \omega_3, \omega_4$. These representations are not equivalent but they are conjugate under the diagram automorphism of order three of $D_4$ (this is essentially the triality principle). Suppose now that the identification of the $D_4$ part of $\mathfrak{l}'_\theta$ with $\mathfrak{o}(8)$ is choosen in such a way that $(D_4, V_{\alpha_1})$ corresponds to the natural representation of $\mathfrak{o}(8)$ in $\mathbb{C}^8$. Then $\omega_4$ is necessarily a *spin* representation of $\mathfrak{o}(8)$. Let $X_{\overline{\alpha}_1}$ be a generic element in $V_{\alpha_1}$. The partial isotropy $(D_4)_{X_{\overline{\alpha}_1}} \simeq \mathfrak{o}(8)_{X_{\overline{\alpha}_1}}$ is well known to be be isomorphic to $\mathfrak{o}(7)$. Hence the partial isotropy representation $((\mathfrak{l}'_\theta)_{X_{\overline{\alpha}_1}}, V_{\alpha_2})$ is isomorphic to $(\mathfrak{o}(7) \times \mathfrak{sl}_2, \omega_4|_{\mathfrak{o}(7)} \otimes \omega_5)$. On the other hand one knows that the restriction of one of the *spin* representation of $\mathfrak{o}(8)$ to $\mathfrak{o}(7)$ is the *spin* representation of $\mathfrak{o}(7)$ (see [5]). The orthogonal of $\overline{\alpha}_1$ in $\mathfrak{h}_\theta$ is a one dimensional space which acts by scalars on $V_{\alpha_2}$ and hence provides the center of $\mathfrak{gl}_2$.

The assertion concerning the relative invariant is then obvious. This ends the proof of (1).

(2) From the obvious embedding $E_7 \hookrightarrow E_8$, the proof of (2) is essentially the same as for (1).

(3) Here $\mathfrak{l}_\theta \simeq D_6 \times \mathfrak{h}_\theta$ where $\dim \mathfrak{h}_\theta = 2$. Let us number the roots as follows:



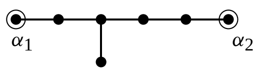

and denote by $(\omega_1, \dots, \omega_6)$ the fundamental weights of $D_6$ corresponding to $(\beta_1, \dots, \beta_6)$. The *PV* $(\mathfrak{l}_\theta, d_1(\theta))$ decomposes into two irreducibles: $d_1(\theta) = V_{\alpha_1} \oplus V_{\alpha_2}$. The highest weight of $V_{\alpha_1}$ is $\omega_1$ and the highest weight of $V_{\alpha_2}$ is $\omega_6$. It is well known that $\omega_6$ is the natural representation of $D_6 \simeq \mathfrak{o}(12)$ on $\mathbb{C}^{12}$. Also the representation $\omega_1$ is well known to be one (among two) *spin* representations of $D_6 \simeq \mathfrak{o}(12)$. Let $X_{\bar{\alpha}_2}$ be a generic element in $V_{\alpha_2}$. The partial isotropy $(\mathfrak{l}'_\theta)_{X_{\bar{\alpha}_2}} \simeq (D_6)_{X_{\bar{\alpha}_2}} \simeq (\mathfrak{o}(12))_{X_{\bar{\alpha}_2}}$ is well known to be isomorphic to $\mathfrak{o}(11)$. On the other hand one knows that the restriction of one of the *spin* representations of $\mathfrak{o}(12)$ to $\mathfrak{o}(11)$ is the *spin* representation of $\mathfrak{o}(11)$ (see [5]). Hence the partial isotropy relatively to $\alpha_2$ is $(\mathfrak{gl}(1) \times \mathfrak{o}(11), \Lambda_1 \otimes \mathit{spin}, V(1) \otimes V(32))$. The assertion concerning the relative invariant is then obvious. $\square$

**Remark 4.4.** It is worth to notice that the preceding three diagrams are admissible, but not $C$-admissible (this will be no longer the case for the diagrams describing the cases (2-1) and (2-2) below). We know from [17] that the corresponding admissible subalgebras are of type $B_2$ in case (1-1), of type $G_2$ in case (1-2), of type $B_2$ in case (1-3).

## 5. The case (2-1)

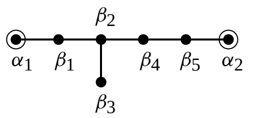
Let us first begin by describing this *PV* as partial isotropy, in a similar way as we did for the cases (1-1), (1-2) and (1-3) in the preceding section.

**Theorem 5.1.** *The PV* $(\mathfrak{gl}(1) \times \mathfrak{o}(9), \Lambda_1 \otimes \mathit{spin}, V(1) \otimes V(16))$ *is the partial isotropy of*




*relatively to* $\alpha_2$. (*Its relative invariant will be built in Corollary* 5.3 *below*).

**Proof.** Here $\mathfrak{l}_\theta \simeq D_5 \times \mathfrak{h}_\theta$ where $\dim \mathfrak{h}_\theta = 2$. Let us number the roots as follows

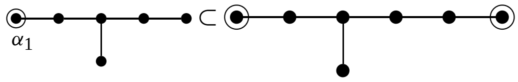



and denote by $(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5)$ the fundamental weights of $D_5$ corresponding to $(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$. The *PV* $(\mathfrak{l}_\theta, d_1(\theta))$ decomposes into two irreducibles: $d_1(\theta) = V_{\alpha_1} \oplus V_{\alpha_2}$. The highest weight of $V_{\alpha_1}$ is $\omega_1$ and the highest weight of $V_{\alpha_2}$ is $\omega_5$ (see rule $(R)$ in Section 2). It is well known that $\omega_5$ is the natural representation of $D_5 \simeq \mathfrak{o}(10)$ on $\mathbb{C}^{10}$. Also the representation $\omega_1$ is well known to be one (among two) *spin* representation of $D_5 \simeq \mathfrak{o}(10)$. Let $X_{\bar{\alpha}_2}$ be a generic element in $V_{\alpha_2}$. The partial isotropy

$$\left(\mathfrak{l}'_\theta\right)_{X_{\bar{\alpha}_2}} \simeq (D_5)_{X_{\bar{\alpha}_2}} \simeq \left(\mathfrak{o}(10)\right)_{X_{\bar{\alpha}_2}}$$

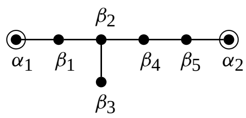
is well known to be isomorphic to $\mathfrak{o}(9)$. On the other hand one knows that the restriction of one of the *spin* representations of $\mathfrak{o}(10)$ to $\mathfrak{o}(9)$ is the *spin* representation of $\mathfrak{o}(9)$ (see [5]). Hence the partial isotropy relatively to $\alpha_2$ is $(\mathfrak{g}l(1) \times \mathfrak{o}(9), \Lambda_1 \otimes \mathit{spin}, V(1) \otimes V(16))$.

Contrary to the cases (1-1), (1-2), and (1-3) the irreducible subdiagram

$D_{\alpha_1} =$ 


is not regular (hence the last diagram is not admissible) and hence the fundamental relative invariant of $(\mathfrak{gl}_1 \times \mathfrak{o}(9), \Lambda_1 \otimes spin, V(1) \otimes V(16))$ is not just the relative invariant of a *PV* with the same space and a "bigger" group. The construction of the relative invariant is more involved and will be given in Corollary 5.3 below. $\square$

We begin by the construction of the relative invariants for the non-irreducible *PV* $(\mathfrak{l}_\theta, d_1(\theta))$ considered in Theorem 5.1 corresponding to the diagram




As we mention just before in the proof of the preceding Theorem, the irreducible subdiagram $D_{\alpha_1}$ is not regular. Hence the corresponding *PV* $(\mathfrak{l}_\theta, d_1(\theta))$ is not the direct sum of two irreducible regular *PV*'s.

However, as a consequence of the classification by Dynkin of the $\mathfrak{sl}_2$-triples in simple Lie algebras, we know that there exists an $\mathfrak{sl}_2$-triple of the form $(Y, H_\theta, X)$ where $X \in d_1(\theta)$, $Y \in d_{-1}(\theta)$. This is exactly told by Dynkin's example $11''$ in Table 19, p. 179 of [6]. Then, by a result of Kostant ([9, Lemma 4.26, p. 990, and remark following the proof], see also [16, Proposition 4.1.4, p. 119]), we know that $(\mathfrak{l}_\theta, d_1(\theta))$ is regular and that its open orbit is the set of elements $X' \in d_1(\theta)$ which can be completed into an $\mathfrak{sl}_2$-triple of the form $(Y', H_\theta, X')$ where $Y' \in d_{-1}(\theta)$. This *PV* hence possesses a non-trivial irreducible relative invariant $P(X_1, X_2)$ depending on the two variables $X_1 \in V_{\alpha_1}$ and $X_2 \in V_{\alpha_2}$. We will now give a description of this polynomial $P$. For this purpose it is easier to work with a group rather than with a Lie algebra. Let $G$ be the adjoint group of $\mathfrak{g}$. Denote by $L_\theta$ the analytical subgroup with Lie algebra $\mathfrak{l}_\theta$ (in fact $L_\theta$ is also the centralizer of $H_\theta$ in $G$). Of course $(L_\theta, d_1(\theta))$ is a *PV* whose derived representation is $(\mathfrak{l}_\theta, d_1(\theta))$. Moreover as the representation $(\mathfrak{l}_\theta, V_{\alpha_1})$ is a *spin* representation (as observed in the proof of Theorem 5.1), the group $L'_\theta$ is isomorphic to $Spin(10)$.

As the highest root of $E_7$ is $\delta = 2\alpha_1 + 2\beta_3 + 3\beta_1 + 4\beta_2 + 3\beta_4 + 2\beta_5 + \alpha_2$, we know that $d_2(\theta) \neq \{0\}$, that $d_{\mathrm{top}}(\theta) = d_3(\theta) \neq \{0\}$ and then $d_p(\theta) = \{0\}$ for $|p| \geqslant 4$. From the root table for $E_7$ (see [1]) we learn in fact that $d_3(\theta) = \mathfrak{g}^{\bar{\delta}} = \mathfrak{g}^{\delta}$ is one dimensional. From the root table we learn also that there exists only one restricted root of $\theta$-height 2, namely $\overline{\alpha_1} + \overline{\alpha_2}$. This implies that $d_2(\theta) = \mathfrak{g}^{\bar{\alpha}_1 + \bar{\alpha}_2}$ is also irreducible under the $L_\theta$-action (see Fig. 2 below).

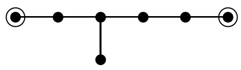
As $d_{\mathrm{top}}(\theta) = d_3(\theta) = \mathfrak{g}^{\delta}$ is one dimensional there exists a character $\chi_\delta$ on $L_\theta$ such that

$$\forall g \in L_\theta, \quad g.X_\delta = \chi_\delta(g) X_\delta$$

where $X_\delta$ is a non-zero root vector in $\mathfrak{g}^{\delta}$. The character $\chi_\delta$ is a natural extension of the root $\delta$ as a character of $L_\theta$.

Let $X_2 \in V_{\alpha_2}$. From elementary facts about the grading defined by the $d_p(\theta)$'s, one knows that $\mathrm{ad}(X_2)|_{V_{\alpha_1}} = \Psi(X_2)$ maps $V_{\alpha_1} = \mathfrak{g}^{\overline{\alpha}_1}$ into $d_2(\theta) = \mathfrak{g}^{\overline{\alpha}_1+\overline{\alpha}_2}$. Also if $X_{-\delta}$ is a non-zero root vector in $\mathfrak{g}^{-\delta} = d_{-\mathrm{top}}(\theta)$, then $\mathrm{ad}(X_{-\delta})|_{d_2(\theta)}$ maps $d_2(\theta)$ into $\mathfrak{g}^{-\overline{\alpha}_1}$ (see Fig. 2 below). Define $\Phi(X_2) = \mathrm{ad}(X_{-\delta}) \circ \Psi(X_2) : V_{\alpha_1} = \mathfrak{g}^{\overline{\alpha}_1} \to \mathfrak{g}^{-\overline{\alpha}_1}$.

**Theorem 5.2.** *Let $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5$ be the fundamental weights of $D_5$ as in the proof of Theorem* 5.1. *Let $B$ be the Killing form on $\mathfrak{g} = E_7$. For $X_1 \in V_{\alpha_1}$ and $X_2 \in V_{\alpha_2}$, define $P(X_1, X_2) = B(X_1, \Phi(X_2)X_1) = B(X_1, [X_{-\delta}, [X_2, X_1]])$. Then $P$ is a non-zero relative invariant of the non-irreducible PV $(L_\theta, d_1(\theta)) \simeq (\mathfrak{o}(10) \times \mathbb{C}^2, \omega_1 \otimes \omega_5, V(16) \times V(10))$ corresponding to the diagram*



*It is irreducible of degree* 3 *and its character is $\chi_\delta$.*

**Proof.** From the definition, if $P$ is non-zero, then certainly it is of degree 3. Moreover we have for $g \in L_\theta$, $X_2 \in V_{\alpha_2}$, $X_1 \in V_{\alpha_1}$:

$$\begin{aligned}\Phi(gX_2).gX_1 &= \big[X_{-\delta}, [gX_2, gX_1]\big] = g\big[g^{-1}X_{-\delta}, [X_2, X_1]\big] \\ &= \chi_\delta(g)g\big[X_{-\delta}, [X_2, X_1]\big] = \chi_\delta(g)g\Phi(X_2)X_1.\end{aligned}$$

Therefore we get:

$$\begin{aligned}P(gX_1, gX_2) &= B\big(gX_1, \Phi(gX_2)gX_1\big) = B\big(gX_1, g\chi_\delta(g)\Phi(X_2)X_1\big) \\ &= \chi_\delta(g)B\big(X_1, \Phi(X_2)X_1\big) = \chi_\delta(g)P(X_1, X_2).\end{aligned}$$

Hence if $P$ is non-zero, then it is a relative invariant of $(L_\theta, d_1(\theta))$ with character $\chi_\delta$.

Let us now prove that effectively we have $P \neq 0$. For a fixed, generic element $X_2 \in V_{\alpha_2}$, and $X_1, X_1' \in V_{\alpha_1}$ we define:

$$C_{X_2}\big(X_1, X_1'\big) = B\big(X_1, \Phi(X_2)X_1'\big).$$

There are two $\mathfrak{sl}_2$-triples at hand: the first one is of the form $(*, *, X_1'' + X_2)$ where $X_1'' \in \mathfrak{g}^{\overline{\alpha}_1}$, the second one is of the form $(X_{-\delta}, H_\delta, X_\delta)$. From elementary properties of $\mathfrak{sl}_2$-modules we know that the map $\mathrm{ad}(X_1'' + X_2) = \mathrm{ad}\, X_2 : \mathfrak{g}^{\overline{\alpha}_1} \to \mathfrak{g}^{\overline{\alpha}_1+\overline{\alpha}_2}$ is onto. But as $\dim \mathfrak{g}^{\overline{\alpha}_1} = \dim \mathfrak{g}^{\overline{\alpha}_1+\overline{\alpha}_2} = 16$, the preceding map is an isomorphism. Similarly the map $\mathrm{ad}\, X_{-\delta} : \mathfrak{g}^{\overline{\alpha}_1+\overline{\alpha}_2} \to \mathfrak{g}^{-\overline{\alpha}_1}$ is injective. Therefore the map

$$\Phi(X_2) = \mathrm{ad}\, X_{-\delta} \circ \mathrm{ad}\, X_2 : V_{\alpha_1} = \mathfrak{g}^{\overline{\alpha}_1} \to \mathfrak{g}^{-\overline{\alpha}_1}$$

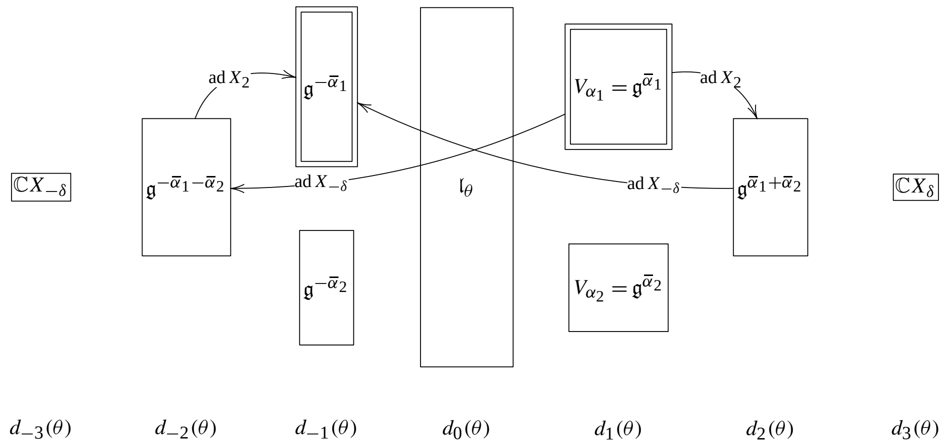



Fig. 2. $E_7$.

is an isomorphism (for $X_2$ generic). Therefore $C_{X_2}$ is a non-degenerate bilinear form on $V_{\alpha_1}$. Hence $P$ is certainly non-zero if $C_{X_2}$ is symmetric. From the invariance and the symmetry of $B$, we see easily that $C_{X_2}$ is symmetric if and only if

$$B\big(X_1, \big[X_{-\delta}, \big[X_2, X_1'\big]\big]\big) = B\big(X_1, \big[X_2, \big[X_{-\delta}, X_1'\big]\big]\big)$$

for $X_2$ generic in $V_{\alpha_2} = \mathfrak{g}^{\overline{\alpha}_2}$, and any $X_1, X_1' \in V_{\alpha_1} = \mathfrak{g}^{\overline{\alpha}_1}$. And this is certainly true if the two maps

$$\begin{matrix} \operatorname{ad} X_{-\delta} \circ \operatorname{ad} X_2 \\ \operatorname{ad} X_2 \circ \operatorname{ad} X_{-\delta} \end{matrix} : V_{\alpha_1} = \mathfrak{g}^{\overline{\alpha}_1} \to \mathfrak{g}^{-\overline{\alpha}_1}$$

coïncide. The graded structure of $E_7$ (relatively to $\theta$), as well as the preceding two maps are summarized in Fig. 2.

We have to prove that for every $X_1 \in V_{\alpha_1} = \mathfrak{g}^{\overline{\alpha}_1}$, we have $[X_{-\delta}, [X_2, X_1]] = [X_2, [X_{-\delta}, X_1]]$. But as $\operatorname{ad} X_\delta$ is injective on $\mathfrak{g}^{-\overline{\alpha}_1}$ (because from the extended diagram of $E_7$, we know that $-\alpha_1(H_\delta) = -1$, and then the injectivity is an elementary property of the $\mathfrak{sl}_2$-triple $(X_{-\delta}, H_\delta, X_\delta)$) it is equivalent to proving that

$$\big[X_\delta, \big[X_{-\delta}, [X_2, X_1]\big]\big] = \big[X_\delta, \big[X_2, [X_{-\delta}, X_1]\big]\big]. \qquad (*)$$

Now we have

$$\big[X_\delta, \big[X_{-\delta}, [X_2, X_1]\big]\big] = \big[-H_\delta, [X_2, X_1]\big] + \big[X_{-\delta}, \big[X_\delta, [X_2, X_1]\big]\big] = \big[-H_\delta, [X_2, X_1]\big].$$

On the other hand we have

$$\begin{aligned} \big[X_\delta, \big[X_2, [X_{-\delta}, X_1]\big]\big] &= \big[X_2, \big[X_\delta, [X_{-\delta}, X_1]\big]\big] = \big[X_2, \big[[X_\delta, X_{-\delta}], X_1\big]\big] \\ &= \big[X_2, [-H_\delta, X_1]\big] = \big[-H_\delta, [X_2, X_1]\big] \end{aligned}$$

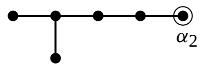
(the last equality is due to the fact that $[H_\delta, X_2] = 0$, which is an easy consequence of the extended Dynkin diagram of $E_7$). Hence the relation $(*)$ above is proved.

We have therefore proved that $P$ is a non-trivial relative invariant with $\chi_\delta$ as character. Let us now prove that $P$ is irreducible.

As the quadratic form

$$X_1 \mapsto P(X_1, X_2) = C_{X_2}(X_1, X_1)$$

is non-degenerate on a 16-dimensional space, it is irreducible. Therefore if $P$ is reducible, then $P = QR$ where $Q$ is quadratic in the $X_1$ variable and where $R$ is linear in the $X_2$ variable. This would imply that the *PV* corresponding to the irreducible subdiagram




(case of a single quadratic form) has a linear form as relative invariant. This is of course not true, hence $P$ is irreducible. $\square$

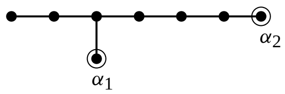
**Corollary 5.3** (notations as before). *The fundamental relative invariant of the partial isotropy PV* $((L_\theta)_{X_2}, V_{\alpha_1})$ (*which is isomorphic to* $(\mathfrak{gl}(1) \times \mathfrak{o}(9), \Lambda_1 \otimes spin, V(1) \otimes V(16))$) *is the quadratic form* $X_1 \mapsto P(X_1, X_2) = B(X_1, [X_2, [X_{-\delta}, X_1]])$. *Its character is the restriction of* $\chi_\delta$ *to* $(L_\theta)_{X_2}$.

**Proof.** It is an immediate consequence of Lemma 4.2. $\square$

**Remark 5.4.** The preceding relative invariant of $(\mathfrak{gl}(1) \times \mathfrak{o}(9), \Lambda_1 \otimes spin, V(1) \otimes V(16))$ was first explicitly described in coordinates by J.I. Igusa [7, Proposition 5, p. 1016]. One can also notice that the non-irreducible *PV* from Theorem 5.2 is "elementary" in the sense of Mortajine [10, p. 142].

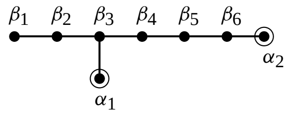
## 6. The case (2-2)

**Theorem 6.1.** *The PV* $(\mathfrak{gl}(1) \times G_2, \Lambda_1 \otimes \Lambda_2, V(1) \otimes V(7))$ *is the partial isotropy of*




*relatively to* $\alpha_1$. (*Its fundamental relative invariant will be built in Corollary* 6.3.)

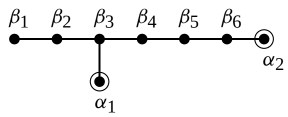
**Proof.** Here $\mathfrak{l}_\theta \simeq A_6 \times \mathfrak{h}_\theta$ where $\dim \mathfrak{h}_\theta = 2$. Let us number the roots as follows:




and denote by $(\omega_1,\dots,\omega_6)$ the fundamental weights of $A_6$ corresponding to $(\beta_1,\dots,\beta_6)$. The $PV$ $\big(\mathfrak{l}_\theta, d_1(\theta)\big)$ decomposes into two irreducibles: $d_1(\theta)=V_{\alpha_1}\oplus V_{\alpha_2}$. The highest weight of $V_{\alpha_1}$ is $\omega_3$ and the highest weight of $V_{\alpha_2}$ is $\omega_6$. The isomorphism $A_6\simeq \mathfrak{sl}(7)$ can be chosen in such a way that the representation $\omega_6$ on $V_{\alpha_2}$ corresponds to the natural representation of $\mathfrak{sl}(7)$ on $\mathbb{C}^7$. Let $X_{\overline{\alpha}_1}$ be a generic element in $V_{\alpha_1}$. The partial isotropy $(\mathfrak{l}'_\theta)_{X_{\overline{\alpha}_1}}\simeq (A_6)_{X_{\overline{\alpha}_1}}\simeq (\mathfrak{sl}(7))_{X_{\overline{\alpha}_1}}$ is known to be isomorphic to $G_2$ (see [22, Example (6), pp. 144–145]). On the other hand one knows that the restriction of the natural representation of $\mathfrak{sl}(7)$ to $G_2$ is the 7 dimensional irreducible representation $\Lambda_2$ of $G_2$. This yields the result.

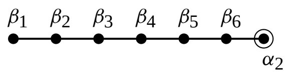
As in the preceding section the construction of the fundamental relative invariant is more involved and will be given below. $\square$

Let us now start the construction of the relative invariant of preceding $PV$. Consider again the following diagram




The irreducible subdiagram

$D_{\alpha_2}$




is not regular. Hence the corresponding $PV$ $(\mathfrak{l}_\theta, d_1(\theta))$ is not the direct sum of two irreducible regular $PV$'s. However, as a consequence of the classification by Dynkin of the $\mathfrak{sl}_2$-triples in simple algebras, we know that there exists an $\mathfrak{sl}_2$-triple of the form $(Y,H_\theta,X)$ where $X\in d_1(\theta)$, $Y\in d_{-1}(\theta)$. This is exactly told by Dynkin's example 32 in Table 20, p. 183 of [6]. Then, again by Kostant's result [9, Lemma 4.26, p. 990, and remark following the proof], see also [16, Proposition 4.1.4, p. 119]), we know that $(\mathfrak{l}_\theta, d_1(\theta))$ is regular and that its open orbit is the set of elements $X'\in d_1(\theta)$ which can be completed into an $\mathfrak{sl}_2$-triple of the form $(Y',H_\theta,X')$, where $Y'\in d_{-1}(\theta)$. This $PV$ possesses a nontrivial irreducible relative invariant $P(X_1,X_2)$ depending on the two variables $X_1\in V_{\alpha_1}$ and $X_2\in V_{\alpha_2}$. We will now give a description of this polynomial $P$. Let $G$ be the adjoint group of $\mathfrak{g}=E_8$. Denote by $L_\theta$ the analytical subgroup of $G$ with Lie algebra $\mathfrak{l}_\theta$ (in fact $L_\theta$ is also the centralizer of $H_\theta$ in $G$). Of course $(L_\theta, d_1(\theta))$ is a $PV$ of parabolic type whose derived representation is $(\mathfrak{l}_\theta, d_1(\theta))$. The representation $(\mathfrak{l}'_\theta, d_1(\theta))$ is isomorphic to $(\mathfrak{sl}_7,\omega_3)$ (rule (R) in Section 2). As the highest root of $E_8$ is

$$\delta = 2\beta_1+4\beta_2+6\beta_3+5\beta_4+4\beta_5+3\beta_6+3\alpha_1+2\alpha_2,$$

we know that $d_{\mathrm{top}}(\theta)=d_5(\theta)$ and hence $d_p(\theta)=0$ for $|p|\geqslant 6$. From the root tables [1] we obtain also the following facts:

- We have $\dim \mathfrak{g}^{\overline{\alpha}_1} = 35$ (this is also the dimension of the representation of weight $\omega_3$ of $\mathfrak{sl}_7$) and $\dim \mathfrak{g}^{\overline{\alpha}_2} = 7$ (this is also the dimension of the natural representation of $\mathfrak{sl}_7$).
- The restricted roots of $\theta$-height 2 are $\overline{\alpha}_1 + \overline{\alpha}_2$ and $2\overline{\alpha}_2$. Moreover $\dim \mathfrak{g}^{\overline{\alpha}_1+\overline{\alpha}_2} = 21$ and $\dim \mathfrak{g}^{2\overline{\alpha}_2} = 7$.
- The only restricted root of $\theta$-height 3 is $2\overline{\alpha}_1 + \overline{\alpha}_2$ and $\dim \mathfrak{g}^{2\overline{\alpha}_1+\overline{\alpha}_2} = 21$.
- The only restricted root of $\theta$-height 4 is $3\overline{\alpha}_1 + \overline{\alpha}_2$ and $\dim \mathfrak{g}^{3\overline{\alpha}_1+\overline{\alpha}_2} = 7$.
- The only restricted root of $\theta$-height 5 is $\overline{\delta} = 3\overline{\alpha}_1 + 2\overline{\alpha}_2$ and $\dim \mathfrak{g}^{\overline{\delta}} = 1$. Hence $\mathfrak{g}^{\overline{\delta}} = \mathfrak{g}^{\delta} = \mathbb{C}X_\delta$.

As $d_{\mathrm{top}}(\theta) = d_5(\theta) = \mathfrak{g}^\delta$ is one dimensional there exists a character $\chi_\delta$ on $L_\theta$ such that

$$\forall g \in L_\theta,\ \forall X_\delta \in \mathfrak{g}^\delta, \quad g.X_\delta = \chi_\delta(g)X_\delta.$$

Let $X_1 \in V_{\alpha_1} = \mathfrak{g}^{\overline{\alpha}_1}$. From elementary facts about the grading defined by the restricted roots, one knows that $(\operatorname{ad} X_1)^3|_{V_{\alpha_2}}$ maps $V_{\alpha_2} = \mathfrak{g}^{\overline{\alpha}_2}$ into $\mathfrak{g}^{3\overline{\alpha}_1+\overline{\alpha}_2} = d_4(\theta)$ (see Fig. 3 below). Also if $X_{-\delta}$ is a non-zero vector in $\mathfrak{g}^{-\delta} = d_{-\mathrm{top}}(\theta)$, then $\operatorname{ad} X_{-\delta}|_{\mathfrak{g}^{3\overline{\alpha}_1+\overline{\alpha}_2}}$ maps $\mathfrak{g}^{3\overline{\alpha}_1+\overline{\alpha}_2}$ into $\mathfrak{g}^{-\overline{\alpha}_2}$. Define

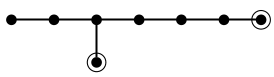
$$\Phi(X_1) = \operatorname{ad} X_{-\delta} \circ (\operatorname{ad} X_1)^3 : \mathfrak{g}^{\overline{\alpha}_2} \to \mathfrak{g}^{-\overline{\alpha}_2}$$

(see Fig. 3 below).

**Theorem 6.2.** *Let $B$ be the Killing form on $\mathfrak{g} = E_8$. For $X_1 \in V_{\alpha_1}$ and $X_2 \in V_{\alpha_2}$, define*

$$P(X_1, X_2) = B\big(X_2, \Phi(X_1)X_2\big) = B\big(X_2, \operatorname{ad} X_{-\delta} \circ (\operatorname{ad} X_1)^3 X_2\big).$$

*Then $P$ is a non-zero relative invariant of the non-irreducible PV $(L_\theta, d_1(\theta)) \simeq (\mathfrak{sl}_7(\mathbb{C}) \times \mathbb{C}^2, \omega_3 \otimes \omega_6, V(35) \times V(7))$ corresponding to the diagram*



*It is irreducible of degree* 5 *and its character is $\chi_\delta$.*

**Proof.** From the definition, if $P$ is non-zero, then certainly it is of degree 5. Moreover for g$\in L_\theta$, $X_1 \in V_{\alpha_1}$, $X_2 \in V_{\alpha_2}$ we have:

$$\begin{aligned}\Phi(gX_1)gX_2 &= \big[X_{-\delta}, \big[gX_1, \big[gX_1, [gX_1, gX_2]\big]\big]\big] = g\big[g^{-1}X_{-\delta}, (\operatorname{ad} X_1)^3 X_2\big] \\ &= g\chi_\delta(g)\Phi(X_1)X_2.\end{aligned}$$

Therefore we get

$$P(X_1, X_2) = B\big(gX_2, \Phi(gX_1)gX_2\big) = B\big(gX_2, gX_{-\delta}(g)\Phi(X_1)X_2\big) = \chi_\delta(g)P(X_1, X_2).$$

Hence if $P$ is non-zero it is a relative invariant of $(L_\theta, d_1(\theta))$ with character $\chi_\delta$. Let us now prove that effectively we have $P \neq 0$. For a fixed generic element $X_1 \in V_{\alpha_1}$, and $X_2, X_2' \in V_{\alpha_2}$ we first define:

$$C_{X_1}\bigl(X_2, X_2'\bigr) = B\bigl(X_2, \Phi(X_1)X_2'\bigr).$$

There are two $\mathfrak{sl}_2$-triples at hand: the first is of the form $(*, *, X_1 + X_2'')$, where $X_2'' \in V_{\alpha_2}$, the second one is of the form $(X_{-\delta}, H_\delta, X_\delta)$. From elementary facts on $\mathfrak{sl}_2$-modules we know that the map

$$\bigl(\operatorname{ad}\bigl(X_1 + X_2''\bigr)\bigr)^3 = (\operatorname{ad} X_1)^3 : \mathfrak{g}^{\overline{\alpha}_2} \to \mathfrak{g}^{3\overline{\alpha}_1+\overline{\alpha}_2}$$

is onto. As $\dim \mathfrak{g}^{\overline{\alpha}_2} = \dim \mathfrak{g}^{3\overline{\alpha}_1+\overline{\alpha}_2} = 7$, the preceding map is an isomorphism. Similarly the map:

$$\operatorname{ad} X_{-\delta} : \mathfrak{g}^{3\overline{\alpha}_1+\overline{\alpha}_2} \to \mathfrak{g}^{-\overline{\alpha}_2}$$

is injective. Therefore the map:

$$\Phi(X_1) = \operatorname{ad} X_{-\delta} \circ (\operatorname{ad} X_1)^3 : V_{\alpha_2} = \mathfrak{g}^{\overline{\alpha}_2} \to \mathfrak{g}^{-\overline{\alpha}_2}$$

is an isomorphism (for $X_1$ generic). Therefore $C_{X_1}$ is a non-degenerate bilinear form on $V_{\alpha_2}$. Hence $P$ is certainly non-zero if $\mathrm{C}_{X_1}$ is symmetric. From the invariance and the symmetry of $B$, we see easily that $C_{X_1}$ is symetric if and only if

$$B\bigl(X_2', \bigl[X_{-\delta}, \bigl[X_1, \bigl[X_1, [X_1, X_2]\bigr]\bigr]\bigr]\bigr) = B\bigl(X_2', \bigl[X_1, \bigl[X_1, \bigl[X_1, [X_{-\delta}, X_2]\bigr]\bigr]\bigr]\bigr)$$

for $X_1$ generic in $V_{\alpha_1} = \mathfrak{g}^{\overline{\alpha}_1}$ and any $X_2, X_2' \in V_{A_2} = \mathfrak{g}^{\overline{\alpha}_2}$. And this is certainly true if the two maps:

$$\begin{matrix} \operatorname{ad} X_{-\delta} \circ (\operatorname{ad} X_1)^3 \\ (\operatorname{ad} X_1)^3 \circ \operatorname{ad} X_{-\delta} \end{matrix} : V_{\alpha_2} = \mathfrak{g}^{\overline{\alpha}_2} \to \mathfrak{g}^{-\overline{\alpha}_2}$$

coincide.

The graded structure of $E_8$ relatively to $\theta$, as well as the preceding two maps are summarized in Fig. 3.

We have to prove that for every $X_2 \in V_{\alpha_2} = \mathfrak{g}^{\overline{\alpha}_2}$ we have:

$$\bigl[X_{-\delta}, \bigl[X_1, \bigl[X_1, [X_1, X_2]\bigr]\bigr]\bigr] = \bigl[X_1, \bigl[X_1, \bigl[X_1, [X_{-\delta}, X_2]\bigr]\bigr]\bigr].$$

As $\alpha_2(H_\delta) < 0$ (just consider the extended diagram of $E_8$ for this), we know that $X_\delta$ is injective on $\mathfrak{g}^{-\overline{\alpha}_2}$. Hence it is equivalent to prove that

$$\bigl[X_\delta, \bigl[X_{-\delta}, \bigl[X_1, \bigl[X_1, [X_1, X_2]\bigr]\bigr]\bigr]\bigr] = \bigl[X_\delta, \bigl[X_1, \bigl[X_1, \bigl[X_1, [X_{-\delta}, X_2]\bigr]\bigr]\bigr]\bigr]. \qquad (**)$$

Now we have:

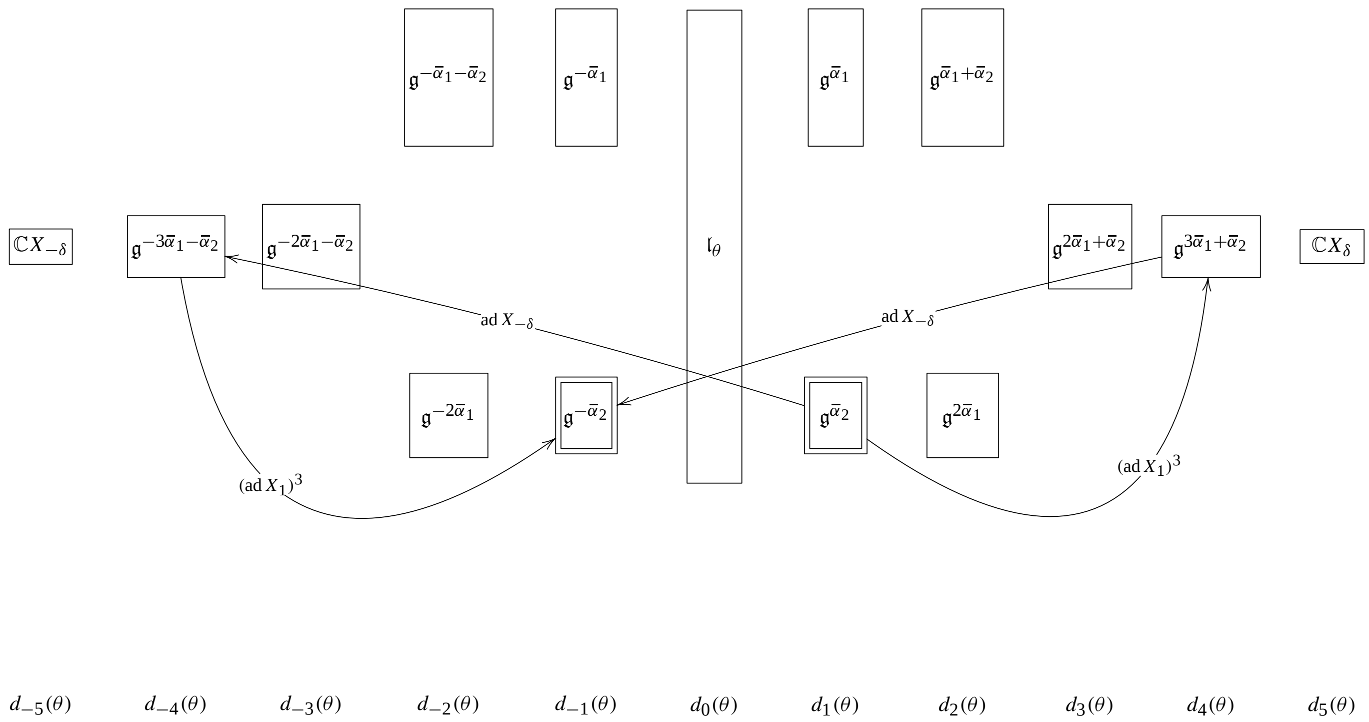

$\mathfrak{g}^{-\bar\alpha_1-\bar\alpha_2}$
$\mathfrak{g}^{-\bar\alpha_1}$
$\mathfrak{g}^{\bar\alpha_1}$
$\mathfrak{g}^{\bar\alpha_1+\bar\alpha_2}$
$\mathbb{C}X_{-\delta}$
$\mathfrak{g}^{-3\bar\alpha_1-\bar\alpha_2}$
$\mathfrak{g}^{-2\bar\alpha_1-\bar\alpha_2}$
$\mathfrak{l}_\theta$
$\mathfrak{g}^{2\bar\alpha_1+\bar\alpha_2}$
$\mathfrak{g}^{3\bar\alpha_1+\bar\alpha_2}$
$\mathbb{C}X_\delta$
ad $X_{-\delta}$
ad $X_{-\delta}$
$\mathfrak{g}^{-2\bar\alpha_1}$
$\mathfrak{g}^{-\bar\alpha_2}$
$\mathfrak{g}^{\bar\alpha_2}$
$\mathfrak{g}^{2\bar\alpha_1}$
$(\mathrm{ad}\,X_1)^3$
$(\mathrm{ad}\,X_1)^3$
$d_{-5}(\theta)$
$d_{-4}(\theta)$
$d_{-3}(\theta)$
$d_{-2}(\theta)$
$d_{-1}(\theta)$
$d_0(\theta)$
$d_1(\theta)$
$d_2(\theta)$
$d_3(\theta)$
$d_4(\theta)$
$d_5(\theta)$

Fig. 3. $E_8$.

$$\begin{aligned}\big[X_\delta,\big[X_{-\delta},\big[X_1,\big[X_1,[X_1,X_2]\big]\big]\big]\big] &= -\big[H_\delta,\big[X_1,\big[X_1,[X_1,X_2]\big]\big]\big] \\ &\quad + \big[X_{-\delta},\underbrace{\big[X_\delta,\big[X_1,\big[X_1,[X_1,X_2]\big]\big]\big]}_{0}\big] \\ &= -\big[H_\delta,\big[X_1,\big[X_1,[X_1,X_2]\big]\big]\big].\end{aligned}$$

On the other hand we have:

$$\begin{aligned}\big[X_\delta,\big[X_1,\big[X_1,\big[X_1,[X_{-\delta},X_2]\big]\big]\big]\big] &= \big[X_1,\big[X_1,\big[X_1,\big[X_\delta,[X_{-\delta},X_2]\big]\big]\big]\big] \\ &= \big[X_1,\big[X_1,\big[X_1,[-H_\delta,X_2]\big]\big]\big] \\ &= -\big[H_\delta,\big[X_1,\big[X_1,[X_1,X_2]\big]\big]\big]\end{aligned}$$

(the last equality is due to the fact that $\alpha_1(H_\delta)=0$, which is an easy consequence of the extended diagram of $E_8$). Hence the relation $(**)$ above is proved.

Therefore $P$ is a non-trivial relative invariant of degree 5 and with $\chi_\delta$ as character.

Let us now prove that $P$ is irreducible. As $\partial^\circ P=5$, if $P$ would not be irreducible we would have $P=QR$ with either

(i) $\partial^\circ Q=1$ and $\partial^\circ R=4$ or
(ii) $\partial^\circ Q=2$ and $\partial^\circ R=3$.

First suppose we are in the (i) case. Then at least one of the two $PV$'s $(L_\theta,V_{\alpha_1})$ and $(L_\theta,V_{\alpha_2})$ would have a relative invariant of degree 1. This is of course not true. Suppose now we are in the (ii) case. Then $Q$ is a non-zero quadratic form on $V_{\alpha_1}\oplus V_{\alpha_2}$ which is relatively invariant under the action of $L_\theta$. Suppose that the restriction of $Q$ to $V_{\alpha_1}$ (respectively $V_{\alpha_2}$) is non-zero, then $(L_\theta,V_{\alpha_1})$ (respectively $(L_\theta,V_{\alpha_2})$) would have a non-zero relative invariant of degree 2, this is known not to be the case. This implies that the partial isotropy $PV$ $((L_\theta)_{X_1},V_{\alpha_2})$ (which is precisely the $PV$ $(\mathfrak{gl}(1)\times G_2,\Lambda_1\otimes\Lambda_2,V(1)\otimes V(7))$) would have a non-zero relative invariant of degree 1, and this again is known not to be true. Hence $P$ is irreducible. □

**Corollary 6.3** (notations as before). *The fundamental relative invariant of the partial isotropy PV* $((L_\theta)_{X_1},V_{\alpha_2})$ *(which is isomorphic to* $(\mathfrak{gl}(1)\times G_2,\Lambda_1\otimes\Lambda_2,V(1)\otimes V(7))$*) is the quadratic form* $X_2\mapsto P(X_1,X_2)=B(X_2,\Phi(X_1)X_2)=B(X_2,\operatorname{ad}X_{-\delta}\circ(\operatorname{ad}X_1)^3X_2)$. *Its character is the restriction of* $\chi_\delta$ *to* $(L_\theta)_{X_1}$.

**Proof.** It is an immediate consequence of Lemma 4.2 and of Theorem 6.2. □

**Remark 6.4.** As $\dim\mathfrak{g}^{\overline{\alpha}_1}=35$, the map

$$\Phi: X_1\mapsto\Phi(X_1)=\operatorname{ad}X_{-\delta}\circ(\operatorname{ad}X_1)^3\in\operatorname{Hom}\big(\mathfrak{g}^{\overline{\alpha}_2},\mathfrak{g}^{-\overline{\alpha}_2}\big)$$

can be identified with the $GL(7)$-equivariant map from $\mathbb{C}^{35}$ into the space of $7\times 7$ symmetric matrices described by Sato and Kimura [22, p. 86]. Note also that the relative invariant described in Corollary 6.3 was first given by Sato and Kimura (see [22]). One can also notice that the non-irreducible *PV* from Theorem 6.2 is again "elementary" in the sense of Mortajine [10, p. 144].

## 7. A general result for *PV*'s of commutative type

In this section we will prove a general result for regular *PV*'s of commutative type which will be used in the next section.

Suppose we are given a regular *PV* of commutative type:

$$\mathfrak{g} = d_{-1}(\theta) \oplus \mathfrak{l}_\theta \oplus d_1(\theta)$$

whith $\theta$ in list $\mathbf{L}_c$. Let $X$ be a generic element in $d_1(\theta)$. We know from Theorem 2.3 that one can put this element into an $\mathfrak{sl}_2$-triple $(Y, H_\theta, X)$ where $Y$ is regular in $d_{-1}(\theta)$. Let $S = (L_\theta)_X = (L_\theta)_Y$ the centralizer of $X$ in $L_\theta$. We can ask for the decomposition of $d_1(\theta)$ into irreducibles under the action of $S$. The result is given in the theorem below.

**Theorem 7.1.** *With the same notations as above, we have*

$$d_1(\theta) = \mathbb{C}X \oplus U_1 \tag{7-1}$$

*where the space $U_1$ is the orthogonal of $\mathbb{C}Y$ in $d_1(\theta)$ with respect to the Killing form of $\mathfrak{g}$. This space is irreducible under $S$ and hence $d_1(\theta)$ decomposes under $S$ as one times the trivial module and one times the module $(S, U_1)$.*

**Proof.** Let $\mathfrak{a}$ be the Lie algebra isomorphic to $\mathfrak{sl}_2$ generated by $(Y, H_\theta, X)$. It is known that if $\mathfrak{s}$ is the Lie algebra of $S$, then $(\mathfrak{a}, \mathfrak{s})$ is a dual pair in $\mathfrak{g}$ (see [18]). Therefore the centralizer of $\mathfrak{s}$ in $d_1(\theta)$, i.e., the isotopic component of the trivial module of $S$, is equal to $\mathbb{C}X$. On the other hand, from its definition, the space $U_1$ is certainly stable under $S$ and decomposition (7-1) is verified. Hence it suffices to prove that the representation $(S, U_1)$ is irreducible.

Suppose that this is not the case. Hence $U_1 = V_1 \oplus W_1$ where $V_1$ and $W_1$ are non-trivial $S$-submodules of $U_1$. And hence the space $d_1(\theta)$ decompose as follows in $S$-submodules:

$$d_1(\theta) = \mathbb{C}X \oplus V_1 \oplus W_1. \tag{7-2}$$

Let $w = \exp(\operatorname{ad} X)\exp(\operatorname{ad} Y)\exp(\operatorname{ad} X)$ be the standard Weyl group element of $\mathfrak{a}$. Then of course $wX = Y$, and if we set $V_{-1} = wV_1$ and $W_{-1} = wW_1$, the following decomposition:

$$d_{-1}(\theta) = \mathbb{C}Y \oplus V_{-1} \oplus W_{-1} \tag{7-3}$$

is $S$-invariant. Let $V_1' = (\mathbb{C}Y \oplus W_{-1})^\perp$ and $W_1' = (\mathbb{C}Y \oplus V_{-1})^\perp$ be the orthogonal (in $d_1(\theta)$) of the corresponding spaces, with respect to the Killing form. Then the decomposition

$$d_1(\theta) = \mathbb{C}X \oplus V_1' \oplus W_1' \tag{7-4}$$

is again $S$-invariant. For $x, y \in d_1(\theta)$ let

$$x = x_0 + x_1 + x_2, \qquad x = x'_0 + x'_1 + x'_2, \qquad y = y'_0 + y'_1 + y'_2$$

be the the decompositions corresponding to (7-2) and (7-4) respectively.

Consider now the following quadratic forms

$$\varphi_0(x) = B\big(wx_0, x'_0\big), \qquad \varphi_1(x) = B\big(wx_1, x'_1\big), \qquad \varphi_2 = B\big(wx_2, x'_2\big).$$

The claim is that these quadratic forms are linearly independent. The preceding quadratic forms are polarizations of the following symmetric bilinear forms on $d_1(\theta)$

$$\Phi_0(x, y) = B\big(wx_0, y'_0\big), \qquad \Phi_1(x, y) = B\big(wx_1, y'_1\big), \qquad \Phi_2(x, y) = B\big(wx_2, y'_2\big).$$

These forms are symmetric because $w^2 = \mathrm{Id}$. Hence it is enough to prove that these bilinear forms are linearly independent. If

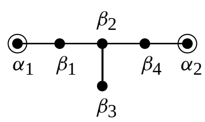
$$\lambda_0\Phi_0 + \lambda_1\Phi_1 + \lambda_2\Phi_2 = 0$$

then, for example $\lambda_1 B(wx_1, y'_1) = 0$ for all $y'_1 \in V'_1$. This implies that $\lambda_1 B(wx_1, y'_1) = 0$ for all $x_1 \in V_1$ and $y'_1 \in V'_1$, and from the definitions this implies that $\lambda_1 = 0$. Similarly one obtains $\lambda_0 = \lambda_2 = 0$ and the claim is proved.

Let $\mathbb{C}^2[d_1(\theta)]^S$ be the space of $S$-invariant quadratic forms on $d_1(\theta)$. From the claim we obtain that $\dim \mathbb{C}^2[d_1(\theta)]^S \geqslant 3$, whereas we know from [19, Théorème 3.5.2] that $\dim \mathbb{C}^2[d_1(\theta)]^S = 2$. Hence $(S, U_1)$ is irreducible. □

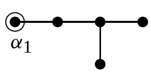
## 8. The case (3-1)

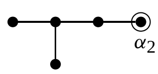
Consider the following diagram:




(8-1)

As the irreducible subdiagrams

$D_{\alpha_1} =$ 

and $D_{\alpha_2} =$ 

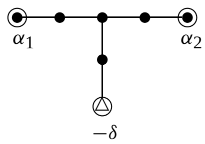

correspond to irreducible regular $PV$ 's of commutative type (i.e., they belong to the list $\mathbf{L}_c$), the diagram (8-1) is $C$-admissible.

Let us denote $(I_{-\overline{\alpha}_1}, H_{\overline{\alpha}_1}, I_{\overline{\alpha}_1})$ and $(I_{-\overline{\alpha}_2}, H_{\overline{\alpha}_2}, I_{\overline{\alpha}_2})$ the two $\mathfrak{sl}_2$-triples which generate the $C$-admissible subalgebra (see Theorem 3.4). It is known from [17] that this $C$-admissible subalgebra of $E_6$ is of type $A_2$. In the sequel of the paragraph we will simply denote by $A_2$ this subalgebra. It is also known from [18] that the centralizer of $A_2$ in $E_6$

is of type $G_2$ (it will simply be noted $G_2$) and that the pair $(A_2, G_2)$ is a dual pair in $E_6$. From the definition it is easy to see that here $G_2$ is simply the centralizer of $I_{\bar\alpha_1} + I_{\bar\alpha_2}$ in $\mathfrak{l}_\theta \simeq D_4 \oplus \mathfrak{h}_\theta$ where $\mathfrak{h}_\theta = \mathbb{C}H_{\bar\alpha_1} \oplus \mathbb{C}H_{\bar\alpha_2}$. In other words $G_2$ is the generic isotropy subalgebra of the *PV* $(\mathfrak{l}_\theta, d_1(\theta))$ corresponding to the diagram (8-1).

There are three non-zero restrictions of positive roots to $\mathfrak{h}_\theta$, namely $\bar\alpha_1, \bar\alpha_2$ and $\bar\alpha_1 + \bar\alpha_2$. We also have $\bar\alpha_1 + \bar\alpha_2 = \bar\delta$ where $\delta = \alpha_1 + 2\beta_1 + 2\beta_3 + 2\beta_4 + 3\beta_2 + \alpha_2$ is the highest root of $E_6$. If one denotes by $\omega_1, \omega_2, \omega_3, \omega_4$ the fundamental weights of $D_4$ corresponding to the set $\beta_1, \beta_2, \beta_3, \beta_4$ of simple positive roots, then the representations $(D_4, \mathfrak{g}^{\bar\alpha_1})$, $(D_4, \mathfrak{g}^{\bar\alpha_2})$ and $(D_4, \mathfrak{g}^{-\bar\alpha_1-\bar\alpha_2} = \mathfrak{g}^{-\bar\delta} = d_1^{-\mathrm{top}}(\theta))$ have dominant weights, with respect to $-\theta$, given respectively by $\omega_1, \omega_4$ and $\omega_3$. They hence realize the three non-equivalent 8-dimensional representations of $D_4$. Recall that these representations are nevertheless exchanged by the outer automorphism of $D_4$ coming from the order 3 automorphism of its Dynkin diagram. This is known to be the background of the triality principle [26].

In a diagrammatical way these representations of $D_4$ (on $\mathfrak{g}^{\bar\alpha_1}$, $\mathfrak{g}^{\bar\alpha_2}$ and $\mathfrak{g}^{\bar\delta}$) can be represented by the following weighted extended diagram of $E_6$:




(8-2)

Here the vertex corresponding to the root $-\delta$ is marked by a triangle $\triangle$ to distinguish it from the simple roots. This diagram contains the same information about the representations as the ordinary (i.e., non-extended) diagram (see Section 2). For example it is clear from this diagram that $(\mathfrak{l}_\theta, \mathfrak{g}^{-\bar\delta})$ is again a regular *PV* of commutative type. Moreover as every irreducible representation of $D_4$ is self-dual, and as the Killing form of $E_6$ realizes non-degenerate dualities between $\mathfrak{g}^{\bar\alpha_1}$ and $\mathfrak{g}^{-\bar\alpha_1}$, $\mathfrak{g}^{\bar\alpha_2}$ and $\mathfrak{g}^{-\bar\alpha_2}$, $\mathfrak{g}^{\bar\delta}$ and $\mathfrak{g}^{-\bar\delta}$, we see that the representations $(D_4, \mathfrak{g}^{\bar\alpha_1})$, $(D_4, \mathfrak{g}^{\bar\alpha_2})$ and $(D_4, \mathfrak{g}^{\bar\delta})$ (whose direct sum is $\mathfrak{n}_\theta^+ = d_1(\theta) \oplus d_2(\theta)$) are the three irreducible 8-dimensional representations of $D_4$.

For further use (Theorem 8.1 below), let us denote by $\chi_{\bar\alpha_1}$ (respectively $\chi_{\bar\alpha_2}$) the character of fundamental relative invariant of the *PV* $(\mathfrak{l}_\theta, \mathfrak{g}^{\bar\alpha_1})$ (respectively $(\mathfrak{l}_\theta, \mathfrak{g}^{\bar\alpha_1})$). Its differential on $\mathfrak{h}_\theta$ is $\bar\alpha_1$ (respectively $\bar\alpha_2$)

Let $I_{\bar\delta} = [I_{\bar\alpha_1}, I_{\bar\alpha_2}]$ and $I_{-\bar\delta} = [I_{-\bar\alpha_1}, I_{-\bar\alpha_2}]$. Then $(I_{-\bar\delta}, H_{\bar\delta} = H_{\bar\alpha_1} + H_{\bar\alpha_2}, I_{\bar\delta})$ is another $\mathfrak{sl}_2$-triple, which can be viewed as a triple attached to the *PV* $(\mathfrak{l}_\theta, \mathfrak{g}^{-\bar\delta})$ by Theorem 2.3. Of course the eight elements $I_{\bar\alpha_1}, I_{\bar\alpha_2}, I_{\bar\delta}, H_{\bar\alpha_1}, H_{\bar\alpha_2}, I_{-\bar\alpha_1}, I_{-\bar\alpha_2}, I_{-\bar\delta}$ are the canonical basis of $A_2$.

Applying now Theorem 7.1 to each of the representations $\mathfrak{g}^{\bar\alpha_1}$, $\mathfrak{g}^{-\bar\alpha_1}$, $\mathfrak{g}^{\bar\alpha_2}$, $\mathfrak{g}^{-\bar\alpha_2}$, $\mathfrak{g}^{\bar\delta}$, $\mathfrak{g}^{-\bar\delta}$ of $D_4$ we obtain six 7-dimensional spaces

$$U_{\pm\bar\alpha_1} \subset \mathfrak{g}^{\pm\bar\alpha_1}, \qquad U_{\pm\bar\alpha_2} \subset \mathfrak{g}^{\pm\bar\alpha_2}, \qquad U_{\pm\bar\delta} \subset \mathfrak{g}^{\pm\bar\delta}$$

such that

$$\mathfrak{g}^{\pm\bar\alpha_1} = \mathbb{C}I_{\pm\bar\alpha_1} \oplus U_{\pm\bar\alpha_1}, \qquad \mathfrak{g}^{\pm\bar\alpha_2} = \mathbb{C}I_{\pm\bar\alpha_2} \oplus U_{\pm\bar\alpha_2}, \qquad \mathfrak{g}^{\pm\bar\delta} = \mathbb{C}I_{\pm\bar\delta} \oplus U_{\pm\bar\delta}$$

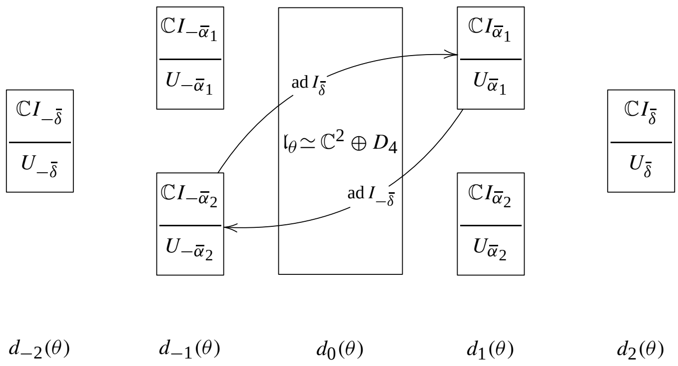



Fig. 4. $E_6$.

and all the representations

$$\big(Z_{\mathfrak{l}_\theta}(I_{\pm\bar\alpha_i}), U_{\pm\bar\alpha_i}\big), \qquad \big(Z_{\mathfrak{l}_\theta}(I_{\pm\bar\delta}), U_{\pm\bar\delta}\big)$$

are irreducible.

Therefore the $A_2$-graded structure of $E_6$ can be represented by Fig. 4.

Suppose now, for example, that we make an identification between $D_4 = \mathfrak{l}'_\theta$ and $\mathfrak{o}(8)$ in such a way that the representation $(D_4, \mathfrak{g}^{\bar\alpha_1})$ corresponds to the "natural" 8-dimensional representation of $\mathfrak{o}(8)$. Then the two others representations $(D_4, \mathfrak{g}^{\bar\alpha_2})$ and $(D_4, \mathfrak{g}^{\bar\delta})$ are the two (half) Spin representations of $\mathfrak{o}(8)$. It is well known (see [22, p. 114]) that the partial isotropy representations $(Z_{\mathfrak{l}_\theta}(I_{\bar\alpha_1}) \simeq \mathfrak{o}(7) \otimes \mathbb{C}, \mathfrak{g}^{\bar\alpha_2})$ and $(Z_{\mathfrak{l}_\theta}(I_{\bar\alpha_1}) \simeq \mathfrak{o}(7) \otimes \mathbb{C}, \mathfrak{g}^{\bar\delta})$ are the Spin representation of $\mathfrak{o}(7)$ (times the corresponding scalar action of $\mathbb{C}$). The generic isotropy of the *PV* $(Z_{\mathfrak{l}_\theta}(I_{\bar\alpha_1}) \simeq \mathfrak{o}(7) \otimes \mathbb{C}, \mathfrak{g}^{\bar\alpha_2})$ has been computed by Sato and Kimura [22, p. 116]. They found that

$$Z_{Z_{\mathfrak{l}_\theta}(I_{\bar\alpha_1})}(I_{\bar\alpha_2}) = Z_{\mathfrak{l}_\theta}(I_{\bar\alpha_1} + I_{\bar\alpha_2}) = G_2.[1] \tag{8-3}$$

Moreover, the representation $(G_2, U_{\bar\alpha_1})$ is the restriction to $G_2$ of the "natural" representation of $\mathfrak{o}(7)$ and hence is still irreducible and is the smallest non-trivial irreducible representation of $G_2$ (see for example [22, pp. 20–21]).

Of course a similar proof shows that the other two representations $(G_2, U_{\bar\alpha_2})$ and $(G_2, U_{\bar\delta})$ are also the same 7-dimensional irreducible representation of $G_2$. From the $A_2$-gradation which is described in Fig. 4, we obtain that

$$\mathrm{ad}\, I_{\bar\delta} : \mathfrak{g}^{-\bar\alpha_1} \to \mathfrak{g}^{\bar\alpha_2}, \qquad \mathrm{ad}\, I_{\bar\delta} : \mathfrak{g}^{-\bar\alpha_2} \to \mathfrak{g}^{\bar\alpha_1},$$
$$\mathrm{ad}\, I_{-\bar\delta} : \mathfrak{g}^{\bar\alpha_1} \to \mathfrak{g}^{-\bar\alpha_2}, \qquad \mathrm{ad}\, I_{-\bar\delta} : \mathfrak{g}^{\bar\alpha_2} \to \mathfrak{g}^{-\bar\alpha_1}$$

(see Fig. 4), and

[1] This proves that the centralizer of $A_2$ is $G_2$ as asserted before.

$$\operatorname{ad} I_{\bar\delta}\big(\mathfrak{g}^{\bar\alpha_1}\big) = \operatorname{ad} I_{\bar\delta}\big(\mathfrak{g}^{\bar\alpha_2}\big) = \{0\},$$
$$\operatorname{ad} I_{-\bar\delta}\big(\mathfrak{g}^{-\bar\alpha_1}\big) = \operatorname{ad} I_{-\bar\delta}\big(\mathfrak{g}^{-\bar\alpha_2}\big) = \{0\}.$$

Moreover, as the preceding linear mappings commute with the action of $G_2$, we obtain that

$$\begin{aligned} &\operatorname{ad} I_{\bar\delta} : U_{-\bar\alpha_1} \to U_{\bar\alpha_2}, \qquad &&\operatorname{ad} I_{\bar\delta} : U_{-\bar\alpha_2} \to U_{\bar\alpha_1}, \\ &\operatorname{ad} I_{-\bar\delta} : U_{\bar\alpha_1} \to U_{-\bar\alpha_2}, \qquad &&\operatorname{ad} I_{-\bar\delta} : U_{\bar\alpha_2} \to U_{-\bar\alpha_1}. \end{aligned} \tag{8-4}$$

Let $A_1^{\mathrm{top}} = \mathbb{C}I_{-\bar\delta} \oplus \mathbb{C}H_{\bar\delta} \oplus \mathbb{C}I_{\bar\delta}$ be the Lie algebra spanned by the $\mathfrak{sl}_2$-triple $(I_{-\bar\delta}, H_{\bar\delta}, I_{\bar\delta})$. To obtain a Lie algebra isomorphic to $\mathfrak{gl}_2$ it suffices to add a one dimensional center to $A_1^{\mathrm{top}}$. But here there is a very natural two dimensional Cartan subalgebra at hand, namely $\mathfrak{h}_\theta = \mathbb{C}H_{\bar\alpha_1} \oplus \mathbb{C}H_{\bar\alpha_2}$. Hence we define

$$\mathfrak{gl}_2^{\mathrm{top}} = A_1^{\mathrm{top}} + \mathfrak{h}_\theta \quad \text{(not a direct sum)}.$$

Let $GL_2^{\mathrm{top}}$ be the subgroup (isomorphic to $GL_2(\mathbb{C})$) of the adjoint group $\operatorname{Ad}(E_6)$ with $\mathfrak{gl}_2^{\mathrm{top}}$ as Lie algebra.

**Theorem 8.1.** *The PV* $(\mathfrak{gl}(2) \times G_2, \Lambda_1 \otimes \Lambda_2, V(2) \otimes V(7))$ ((3-1) *in Table* 1) *is realized infinitesimally inside* $E_6$ *as*

$$\big(\mathfrak{gl}_2^{\mathrm{top}} \times G_2, U_{\bar\alpha_1} \times U_{-\bar\alpha_2}\big).$$

*Let $B$ be the Killing form of $E_6$. For $X \in U_{\bar\alpha_1}$ and $Y \in U_{-\bar\alpha_2}$, the polynomial*

$$P(X,Y) = B\big(\big[X, (\operatorname{ad} I_{\bar\alpha_2})^2 Y\big], \big[(\operatorname{ad} I_{-\bar\alpha_1})^2 X, Y\big]\big)$$

*is the fundamental relative invariant of this PV. The character of $P$ is $\chi_{\bar\alpha_1}^2 \chi_{\bar\alpha_2}^{-2}$.*

**Proof.** The first statement is an immediate consequence of formulas (8-4).

Before we prove the second statement, let us explain the construction of $P$ by using Fig. 5 (where $Y' = (\operatorname{ad} I_{\bar\alpha_2})^2(Y)$, $X' = (\operatorname{ad} I_{-\bar\alpha_1})^2(X)$, $Z = [X, Y']$, $Z' = [X', Y]$ and $P(X,Y) = B(Z, Z')$).

Let us first prove that $P$ is invariant under $\exp(A_1^{\mathrm{top}} \times G_2) \subset \operatorname{Ad}(E_6)$. Let $g \in \exp(G_2) \subset \operatorname{Ad}(E_6)$. Then

$$\begin{aligned} P(gX, gY) &= B\big(\big[gX, (\operatorname{ad} I_{\bar\alpha_2})^2(gY)\big], \big[(\operatorname{ad} I_{-\bar\alpha_1})^2 gX, gY\big]\big) \\ &= B\big(g\big[X, (\operatorname{ad} I_{\bar\alpha_2})^2(Y)\big], g\big[(\operatorname{ad} I_{-\bar\alpha_1})^2(X), Y\big]\big) \\ &\qquad \text{(because } (A_2, G_2) \text{ is a dual pair in } E_6) \\ &= B\big(\big[X, (\operatorname{ad} I_{\bar\alpha_2})^2(Y)\big], \big[(\operatorname{ad} I_{-\bar\alpha_1})^2(X), Y\big]\big) = P(X,Y). \end{aligned}$$

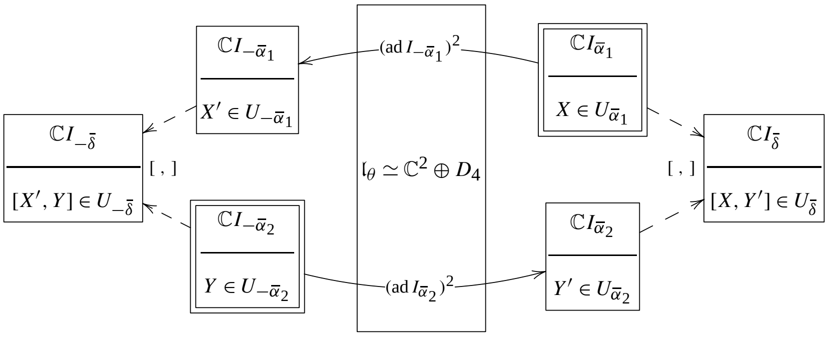



Fig. 5. $E_6$.

Hence $P$ is $\exp(G_2)$-invariant. In order to prove that $P$ is also invariant under $\exp(A_1^{\mathrm{top}})$ it is enough to prove that it is invariant under $\exp(t \operatorname{ad}(I_{\bar{\delta}}))$ and $\exp(t \operatorname{ad}(I_{-\bar{\delta}}))$, $t \in \mathbb{R}$.

For $X \in U_{\bar{\alpha}_1}$, we have $\exp(t \operatorname{ad}(I_{\bar{\delta}}))(X) = X$ (because $d_3(\theta) = \{0\}$).

For $Y \in U_{-\bar{\alpha}_2}$, we have $\exp(t \operatorname{ad}(I_{\bar{\delta}}))(Y) = Y + t[I_{\bar{\delta}}, Y]$ and we note that $[I_{\bar{\delta}}, Y] \in U_{\bar{\alpha}_1}$. Then we obtain $(\operatorname{ad} I_{\bar{\alpha}_2})^2 \exp(t \operatorname{ad}(I_{\bar{\delta}}))(Y) = (\operatorname{ad} I_{\bar{\alpha}_2})^2(Y)$.

Therefore

$$\left[\exp\left(t \operatorname{ad}(I_{\bar{\delta}})\right)(X), (\operatorname{ad} I_{\bar{\alpha}_2})^2 \exp\left(t \operatorname{ad}(I_{\bar{\delta}})\right)(Y)\right] = \left[X, (\operatorname{ad} I_{\bar{\alpha}_2})^2(Y)\right] \in d_2(\theta) = \mathfrak{g}^{\bar{\delta}}.$$

Similarly one obtains

$$\begin{aligned}
&\left[(\operatorname{ad} I_{-\bar{\alpha}_1})^2 \exp\left(t \operatorname{ad}(I_{\bar{\delta}})\right)(X), \exp\left(t \operatorname{ad}(I_{\bar{\delta}})\right)(Y)\right] \\
&\quad = \left[(\operatorname{ad} I_{-\bar{\alpha}_1})^2(X), Y + t[I_{\bar{\delta}}, Y]\right] \\
&\quad = \underbrace{\left[(\operatorname{ad} I_{-\bar{\alpha}_1})^2(X), Y\right]}_{\in \mathfrak{g}^{-\bar{\delta}}} + t \underbrace{\left[(\operatorname{ad} I_{-\bar{\alpha}_1})^2(X), [I_{\bar{\delta}}, Y]\right]}_{\in \mathfrak{l}_\theta}.
\end{aligned}$$

As $\mathfrak{l}_\theta$ and $\mathfrak{g}^{\bar{\delta}}$ are orthogonal under $B$ we get:

$$\begin{aligned}
&P\left(\exp\left(t \operatorname{ad}(I_{\bar{\delta}})\right)(X), \exp\left(t \operatorname{ad}(I_{\bar{\delta}})\right)(Y)\right) \\
&\quad = B\left(\left[X, (\operatorname{ad} I_{\bar{\alpha}_2})^2(Y)\right], \left[(\operatorname{ad} I_{-\bar{\alpha}_1})^2(X), Y\right]\right) = P(X, Y).
\end{aligned}$$

A similar proof shows that $P$ is also invariant under $\exp(t \operatorname{ad}(I_{-\bar{\delta}}))$, therefore $P$ is invariant under $\exp(A_1^{\mathrm{top}})$ and hence under $\exp(A_1^{\mathrm{top}} \times G_2)$.

From the definition it is also clear that if $h \in \exp(\operatorname{ad} \mathfrak{h}_\theta)$, then

$$P(hX, hY) = \chi_{\bar{\alpha}_1}^2(h)\chi_{\bar{\alpha}_2}^{-2}(h)P(X, Y).$$

Therefore if $P \neq 0$ it is a relative invariant of $GL_2^{\mathrm{top}}$ with the announced character.

Now we will prove that $P$ is non-zero. We choose first a Chevalley system for $(E_6, \mathfrak{h})$, in the sense of [2, Chapter 8, §2, $n°4$, Definition 3]. Recall that in particuliar this is a family of root vectors $(X_\alpha)_{\alpha\in\Sigma}$ such that $(X_{-\alpha}, H_\alpha, X_\alpha)$ is a $\mathfrak{sl}_2$-triple and such that for $\alpha, \beta \in \Sigma$, we have $[X_\alpha, X_\beta] = N_{\alpha,\beta}X_{\alpha+\beta}$, where $N_{\alpha,\beta} \in \mathbb{Z}$ and $N_{-\alpha,-\beta} = N_{\alpha,\beta}$. As the $PV$ $(\mathfrak{l}_\theta, \mathfrak{g}^{\overline{\alpha}_1})$ is of commutative type $D_{\alpha_1}$ (see (8-2)), we deduce from [11], that it is of "rank 2" and that we can choose a regular the element $I_{\overline{\alpha}_1}$ to be equal to $X_{\alpha_1} + X_{\gamma_1}$ where $\gamma_1 = \alpha_1 + 2\beta_1 + 2\beta_2 + \beta_3 + \beta_4$ is the highest root of the embedded $D_5$ generated by $\alpha_1, \beta_1, \beta_2, \beta_3, \beta_4$ (see (8-1)). Moreover the roots $\alpha_1$ and $\gamma_1$ are strongly orthogonal. Similarly one can choose $I_{\overline{\alpha}_2} = X_{\alpha_2} + X_{\gamma_2}$ where $\gamma_2 = \beta_1 + \beta_3 + 2\beta_2 + 2\beta_4 + \alpha_2$ is the highest root of the embedded $D_5$ generated by $\beta_1, \beta_2, \beta_3, \beta_4, \alpha_2$ (see $(8-1)$). Again the roots $\alpha_2$ and $\gamma_2$ are strongly orthogonal. On the negative side we choose $I_{-\overline{\alpha}_1} = X_{-\alpha_1} + X_{-\gamma_1}$ and $I_{-\overline{\alpha}_2} = X_{-\alpha_2} + X_{-\gamma_2}$. Then $X = X_{\alpha_1} - X_{\gamma_1}$ is also generic in $\mathfrak{g}^{\overline{\alpha}_1}$. Moreover as $B(X_{\alpha_1} - X_{\gamma_1}, I_{-\overline{\alpha}_1}) = B(X_{\alpha_1} - X_{\gamma_1}, X_{\alpha_1} + X_{\gamma_1}) = 0$, we have in fact $X \in U_{\overline{\alpha}_1}$. In a similar way we prove that $Y = X_{-\alpha_2} - X_{-\gamma_2}$ belongs to $U_{-\overline{\alpha}_2}$ (Y is also generic in $\mathfrak{g}^{-\overline{\alpha}_2}$). Easy computations show also that

$$X' = (\operatorname{ad} I_{-\overline{\alpha}_1})^2 X = 2(X_{-\alpha_1} - X_{-\gamma_1}) \quad \text{and} \quad Y' = (\operatorname{ad} I_{\overline{\alpha}_2})^2 Y = 2(X_{\alpha_2} - X_{\gamma_2}).$$

Then we have:

$$\begin{aligned}
&P(X,Y)\\
&\quad = B\big([X, Y'], [Y, X']\big)\\
&\quad = B\big(\big[X_{\alpha_1} - X_{\gamma_1}, 2(X_{\alpha_2} - X_{\gamma_2})\big], \big[X_{-\alpha_2} - X_{-\gamma_2}, 2(X_{-\alpha_1} - X_{-\gamma_1})\big]\big)\\
&\quad = 4B\big([X_{\alpha_1}, -X_{\gamma_2}] + [-X_{\gamma_1}, X_{\alpha_2}], [X_{-\alpha_1}, X_{-\gamma_2}] + [X_{-\alpha_2}, -X_{-\gamma_1}]\big)\\
&\quad = 4B(-N_{\alpha_1,\gamma_2}X_{\alpha_1+\gamma_2} + N_{\alpha_2,\gamma_1}X_{\alpha_2+\gamma_1}, N_{\alpha_1,\gamma_2}X_{-\alpha_1-\gamma_2} - N_{\alpha_2,\gamma_1}X_{-\alpha_2-\gamma_1})\\
&\quad = -4(N_{\alpha_1,\gamma_2})^2 B(X_{\alpha_1+\gamma_2}, X_{-\alpha_1-\gamma_2}) - 4(N_{\alpha_2,\gamma_1})^2 B(X_{\alpha_2+\gamma_1}, X_{-\alpha_2-\gamma_1})\\
&\quad = 2(N_{\alpha_1,\gamma_2})^2 B(H_{\alpha_1+\gamma_2}, H_{\alpha_1+\gamma_2}) + 2(N_{\alpha_2,\gamma_1})^2 B(H_{\alpha_2+\gamma_1}, H_{\alpha_2+\gamma_1}) > 0.
\end{aligned}$$

Therefore $P$ is a non-zero polynomial.

As we know from [22, §7, I, Exemple (26), p. 147] that this $PV$ has a unique irreducible relative invariant of degree 4, the polynomial $P$ is irreducible. □

**Remark 8.2.** There exists another connection between the $PV$'s considered in this paper and $PV$'s of parabolic type. The $PV$'s from Table 1 can be considered as sub$PV$'s of $PV$'s of parabolic type through the following embeddings:

(1-1) $\mathfrak{gl}(2) \times Spin(7) \to \mathfrak{gl}(2) \times \mathfrak{o}(8)$.
(1-2) $\mathfrak{gl}(3) \times Spin(7) \to \mathfrak{gl}(3) \times \mathfrak{o}(8)$.
(1-3) $\mathfrak{gl}(1) \times Spin(11) \to \mathfrak{gl}(1) \times Spin(12)$.
(2-1) $\mathfrak{gl}(1) \times Spin(9) \to \mathfrak{gl}(1) \times \mathfrak{o}(16)$.
(2-1) $\mathfrak{gl}(1) \times G_2 \to \mathfrak{gl}(1) \times \mathfrak{o}(7)$.
(3-1) $\mathfrak{gl}(2) \times G_2 \to \mathfrak{gl}(2) \times \mathfrak{o}(7)$.

(See [22, (17), (18), (22), (23), (25), (26), p. 146], for the fact that the *PV*'s of the right hand side are of parabolic type see [14] or [15]).

The author thanks the referee for suggesting this remark.

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